Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods

Fall 2017

Exercise set 6

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Show that $(\mathbb{C}^n, \|.\|_2))$ is complete.

2 Show that $(\ell^2, \|.\|_2)$ is a Banach space.

3 Let $(X, \|.\|)$ be a normed space.

a) Show that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X.

b) Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $(y_n)_{n\in\mathbb{N}}$ another sequence in X. If we have for any $\varepsilon > 0$ that there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$

$$\|y_n - y_m\| \leq \|x_n - x_m\|,$$

then $(y_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X.

4 Prove the following two statements for a normed space $(X, \|.\|)$.

a) Any ball $B_r(x) = \{y \in X : \|x - y\| < r\}$ in $(X, \|.\|)$ is bounded and $\mathrm{diam}(B_r(x)) \leq 2r$.

b) If A is a bounded subset of $(X, \|.\|)$, then for any $a \in A$ we have $A \subseteq B_{\mathrm{diam}(A)}(a)$.

5 **a)** Let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

in $C[a, b]$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$.

b) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence on $[0,1]$ defined by $f_n(x) = \frac{1}{1+nx}$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.

6 Let f be a Lipschitz function $f : (X, \|.\|_X) \to (Y, \|.\|_Y)$. Show that f is continuous.