Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2017

Exercise set 5

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a normed space $(X, \|.\|)$.

a) Show that $(x_n)_{n\in\mathbb{N}}$ is a bounded subset of X.

b) Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

2 We denote by c_f the vector space of all sequences with only finitely many non-zero terms. Show that c_f is not a Banach space.

3 For each $n \in \mathbb{N}$, let

$$x^{(n)} := (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots),$$

which we regard as an element of the space $\ell^p(\mathbb{R})$ (for any given $p \in [1, \infty]$).

a) Find the limit of the sequence $\left(x^{(n)}\right)_{n\geq 1}$ in $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$. Prove your claim.

b) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$? If the limit exists, find it and prove that it is the limit.

c) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$? If the limit exists, find it and prove that it is the limit.

4 Let $C[a, b]$ be the vector space of all continuous functions $f\colon [a, b] \to \mathbb{R}$.
We will consider two norms on this space, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

a) Prove that for all $f \in C[a, b]$ we have

$$\|f\|_1 \leq (b - a)\,\|f\|_\infty .$$

b) Let (f_n) be a sequence in $C[a, b]$.
Prove that if $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $f_n \to f$ with respect to $\|\cdot\|_1$.

c) Show that the reverse of the statement in b) is not always true.