# TMA4145 – Linear Methods
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# Introduction

The goal of this course is to present basic facts about vector spaces and mappings between vector spaces in a form suitable for engineers, scientists and mathematicians. The presentation is addressed to students with variying backgrounds.

A special emphasis is put towards general methods and on abstract reasoning. The material in this course is supposed to prepare you for the advanced courses in your respective study program. You might encounter for the first time rigorous reasoning and there will be a particular focus on definitions, statements (=lemmas, propositions, theorems) and proofs.

In the first chapter we discuss basic notions such as sets, functions and the cardinality of a set.

These notes are accompanying the course TMA4145 Linear methods.

CHAPTER 1

# Sets and functions

Basic definitions and theorems about sets and functions are the content of this chapter and are presented in the setting of Naive Set Theory. These notions set the stage for turning our intuition about collections of objects and relations between these objects.

## 1.1. Sets

DEFINITION 1.1.1. A *set* is a collection of distinct objects, its *elements*. If an object $x$ is an element of a set $X$, we denote it by $x \in X$. If $x$ is not an element of A, then we write $x \notin X$.

A set is uniquely determined by its elements. Suppose $X$ and $Y$ are sets. Then they are identical, $X = Y$, if they have the same elements. More formalized, $X = Y$ if and only if for all $x \in X$ we have $x \in Y$, and for all $y \in Y$ we have $y \in X$.

DEFINITION 1.1.2. Suppose $X$ and $Y$ are sets. Then $Y$ is a subset of $X$, denoted by $Y \subset X$, if for all $y \in Y$ we have $y \in X$.

If $Y \subseteq X$, one says that $Y$ is contained in $X$. If $Y \subseteq X$ and $X \neq Y$, then $Y$ is a proper subset of $X$ and we use the notation $Y \subset X$. The most direct way to prove that two sets $X$ and $Y$ are equal is to show that

$$x \in X \iff x \in Y$$

for any element $x$. (Another way is to prove a double inclusion: if $x \in X$ then $x \in Y$, establishing that $X \subset Y$ and if $x \in Y$, then $x \in X$, establishing that $Y \subset X$.)

The *empty set* is a set with no elements, denoted by $\emptyset$.

PROPOSITION 1.1.3. *There is only one empty set.*

PROOF. Suppose $E_1$ and $E_2$ are two empty sets. Then for all elements $x$ we have that $x \notin E_1$ and $x \notin E_2$. Hence $E_1 = E_2$. □

Some familiar sets are given by the various number systems:

(1) $\mathbb{N} = \{1, 2, 3, ...\}$ the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$;
(2) $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ the set of integerr;

(3) $\mathbb{Q} = \{p/q : \ p, q \in \mathbb{Z}\}$ the set of rational numbers;
(4) $\mathbb{R}$ denotes the set of real numbers;
(5) $\mathbb{C}$ denotes the set of complex numbers.

For real numbers $a, b$ with $a < b < \infty$ we denote by $[a, b]$ the closed bounded interval, and by $(a, b)$ the open bounded interval. The length of these bounded intervals is $b - a$.
Here are a few constructions related to sets.

DEFINITION 1.1.4. Let $X$ and $Y$ be sets.

- The *union* of $X$ and $Y$, denoted by $X \cup Y$, is defined by

$$X \cup Y = \{z | \, z \in X \quad \text{or } z \in Y\}.$$

- The *intersection* of $X$ and $Y$, denoted by $X \cap Y$, is defined by

$$X \cap Y = \{z | \, z \in X \quad \text{and } z \in Y\}.$$

- . The *difference set* of $X$ from $Y$, denoted by $X \backslash Y$, is defined by

$$X \backslash Y = \{z \in X : \ z \in X \quad and \ z \neq Y\}.$$

  If all sets are contained in one set $X$, then the difference set $X \backslash Y$ is called the *complement* of $Y$ and denoted by $Y^c$.
- The *Cartesian product* of $X$ and $Y$, denoted by $X \times Y$, is the set

$$X \times Y = \{(x, y) | \, x \in X, y \in Y\},$$

  i.e the set of all ordered pairs $(x, y)$, with $x \in X$ and $y \in Y$. Recall an ordered pair has the property that $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.
- $\mathcal{P}(X)$ denotes the set of all subsets of $X$.

Here are some basic properties of sets.

LEMMA 1.1. *Let $X, Y$ and $Z$ be sets.*

(1) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ *and* $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ *(distribution law)*
(2) $(X \cup Y)^c = X^c \cap Y^c$ *and* $(X \cap Y)^c = X^c \cup Y^c$ *(de Morgan's laws)*
(3) $X \backslash (Y \cup Z) = (X \backslash Y) \cap (X \backslash Z)$ *and* $X \backslash (Y \cap Z) = (X \backslash Y) \cup (X \backslash Z)$
(4) $(X^c)^c = X$.

PROOF. (1) Let us prove one of de Morgan's relations. Let us use the most direct approach. Keep in mind that $x \in E^c \iff$

$x \notin E$. We then have:

$$x \in (X \cup Y)^c \iff x \notin X \cup Y \iff x \notin X \textbf{ and } x \notin Y$$
$$\iff x \in X^c \text{ and } x \in Y^c \iff x \in X^c \cap Y^c.$$

This proves the identity.

(2)
$$x \in (X^c)^c \iff x \notin X^c \iff x \in X.$$

□

Note that if you have a statement involving ∪ and ∩. Then you get another true statement if you interchange ∪ with ∩ and ∩ with ∪, as one can see in the lemma. This is part of the field Boolean algebra.

## 1.2. Functions

Let $X$ and $Y$ be sets. A function with *domain* $X$ and *codomain* $Y$, denoted by $f : X \to Y$, is a relation between the elements of $X$ and $Y$ satisfying the properties: for all $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$, we denote it by: $f(x) = y$.

By definition, for each $x \in X$ there is exactly one $y \in Y$ such that $f(x) = y$. We say that $y$ the *image* of $x$ under $f$. The *graph* $G(f)$ of a function $f$ is the subset of $X \times Y$ defined by

$$G(f) = \{(x, f(x)) |\, x \in X\}.$$

The *range* of a function $f : X \to Y$, denoted by range($f$), or $f(X)$, is the set of all $y \in Y$ that are the image of some $x \in X$:

$$\text{range}(f) = \{y \in Y |\, \text{there exists}\, x \in X \,\text{such that}\, f(x) = y\}.$$

The *pre-image* of $y \in Y$ is the subset of all $x \in X$ that have $y$ as their image. This subset is often denoted by $f^{-1}(y)$:

$$f^{-1}(y) = \{x \in X |\, f(x) = y\}.$$

Note that $f^{-1}(y) = \emptyset$ if and only if $y \in Y \backslash \text{ran}(f)$.

Here are some simple examples of functions.

$$|x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| = \max\{x, -x\}$. We define the positive, $x^+$ and negative part, $x^-$ of $x \in \mathbb{R}$:

$$x^+ = \max\{x, 0\}, \quad \text{and} \quad x^- = \max -x, 0,$$

so we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

The following notions are central for the theory of functions.

DEFINITION 1.2.1. Let $f : X \to Y$ be a function.

(1) We call $f$ *injective* or one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e. no two elements of the domain have the same image. Equivalently, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
(2) We call $f$ *surjective* or *onto* if $\mathrm{ran}(f) = Y$, i.e. each $y \in Y$ is the image of at least one $x \in X$.
(3) We call $f$ *bijective* if $f$ is both injective and surjective.

Note that a bijective function matches up the elements of $X$ with those of $Y$ so that in some sense these two sets have the same number of elements.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions so that the range of $f$ coincides with the domain of g. Then we define the *composition*, denoted by $g \circ f$, as the function $g \circ f : X \to Z$, defined by $x \mapsto g(f(x))$.

For every set X, we define the *identity map*, denoted by $\mathrm{id}_X$ or id where $\mathrm{id}(x) = x$ for all $x \in X$.

LEMMA 1.2. *Let $f : X \to Y$ and $g : Y \to Z$ be two bijections. Then $g \circ f$ is also a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

LEMMA 1.3. *Let $f\colon X \to Y$ be a function and let $C, D \subset Y$. Then*

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

PROOF.

$$\begin{aligned} x \in f^{-1}(C \cup D) &\iff f(x) \in C \cup D \iff f(x) \in C \text{ or } f(x) \in D \\ &\iff x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \iff x \in f^{-1}(C) \cup f^{-1}(D). \end{aligned}$$

□

If one has a function $f$ that maps elements in $X$ to $Y$, then it is often desirable to reverse this assignment. Let us introduce some notions to address this basic problem.

DEFINITION 1.2.2. Let $f$ be a function from $X$ to $Y$.

- The mapping $f$ is said to be *left invertible* if there exists a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. We call $g$ a *left inverse* of $f$ and denote it by $f_l^{-1}$.
- The mapping $f$ is said to be *right invertible* if there exists a function $h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$. We call $h$ a *right inverse* of $f$ and denote it by $f_r^{-1}$.

- The mapping $f$ is said to be *invertible* if there exists a $g : Y \to X$ such that $g \circ f = f \circ g = \mathrm{id}$, the so-called *inverse* of $f$ and denoted by $f^{-1}$.

One may think of a left and right inverse in layman terms: (i) If you map an element of the domain via a function to an element in the target space, then the left inverse tells you how to go back to where you started from;(ii) If one wants to get to a point in the target, then the right inverse tells you a possible place to start in the domain. The inverse of a function has some important properties.

LEMMA 1.4. *Given an invertible function $f : X \to Y$.*

(1) *The inverse function $f^{-1} : Y \to X$ is unique.*
(2) *The inverse function is also invertible and we have $(f^{-1})^{-1} = f$.*

PROOF. (1) Suppose there are two inverse functions $g_i : Y \to X$, $i = 1, 2$. By assumption we have that $f \circ g_1 = \mathrm{id}$ and $g_2 \circ f = \mathrm{id}$. Hence we have

$$g_2(y) = g_2(fg_1(y)) = g_2f(g_1(y)) = g_1(y) \quad \text{for all } y \in Y,$$

i.e. $g_1 = g_2$.

(2) Exercise.

$\square$

Let us give a description of left, right invertibility and invertibility in more concrete terms.

PROPOSITION 1.2.3. *Given a function $f : X \to Y$.*

(1) *$f$ is left invertible if and only if it is injective.*
(2) *$f$ is right invertible if and only if it is surjective.*
(3) *$f$ is invertible if and only if it is injective and surjective, i.e. if $f$ is bijective.*

PROOF. (1) Let us assume that $f$ is injective. Then $f : x \to \mathrm{ran}(f)$ is invertible with $f^{-1} : \mathrm{ran}(f) \to X$. Let $g : Y \to X$ be any extension of this inverse. Then $g \circ f = \mathrm{id}_X$.
Suppose $f$ is left invertible. Assume there are $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$. Then

$$x_1 = f_l^{-1}(f(x_1)) = f_l^{-1}(f(x_2)) = x_2,$$

i.e. $f$ is injective.

(2) Let us assume that $f$ is surjective. Pick an arbitrary element $z \in Y$, wich is by assumption an element of $\mathrm{ran}(f)$. Hence $z$ has at least one pre-image in $X$ and thus $f^{-1}(z) \neq \emptyset$.Take

$y_1 \neq y_2$. Then the sets $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ in $X$ are disjoint. Let us pick from each set $f^{-1}(\{y\})$ an element $x$ and define $x := h(y)$. Then $h : Y \to X$ and $f \circ h = \mathrm{id}_Y$.
Suppose that $f$ is right invertible. Then we have for $y \in Y$ that $f(f_r^{-1})(y) = f(x)$ where we set $x$ to be $x = f_r^{-1}(y)$. In other words, $y$ is in the range of $f$.

(3) Follows from the other assertions.

□

A consequence of the characterizations of left and right invertibility is the observation:

Remark 1.2.4. If $f : X \to Y$ is left invertible such that $\mathrm{ran}(f) \neq Y$, then there are many left inverses. However the restriction of any left inverse of $f$ to $\mathrm{ran}(f)$ is unique.
One the other hand if $f : X \to Y$ is right invertible such that $f$ is surjective but not injective, then $f$ will have many right inverses.

Our study of linear mappings will provide ample examples of the aforementioned notions. Here we just give one example.

Example 1.2.5. Given the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T = Ax$ with

$$A = \begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix}.$$

Then the matrix

$$A_l^{-1} = \frac{1}{9} \begin{pmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{pmatrix}$$

induces a left inverse $T_l^{-1}$ of $T$.
This left inverse is not unique, for example

$$\frac{1}{2} \begin{pmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{pmatrix}$$

also gives a left inverse. One can turn this example into one for right inverses as well, see problem set 1.

## 1.3. Cardinality of sets

Bijective functions provide us with a way to compare the size of two sets. We start with the case of finite sets.

Definition 1.3.1. Two sets $X$ and $Y$ have equal cardinality, if there is a bijective map $f : X \to Y$. If there is an injective map from $X$ to $Y$, then we say that the cardinality of $X$ is less than or equal to

the cardinality of $Y$.
A set $X$ has n elements if there is a bijection between $X$ and the set $\{0, 1, ..., n-1\}$. We denote the set $\{0, 1, ..., n-1\}$ by $n$. A set $X$ is countable if there is a bijection with $\mathbb{N}$. In other words, $X$ is countable if we can arrange its elements in an infinite sequence $\{x_1, x_2, x_3, ...\}$ such that eqch element occurs exactly once in the sequence.

REMARK 1.3.2. There is some more terminology that we will not use in the course. A set $X$ is called at most countable if there is an injection from $X$ to $\mathbb{N}$.

EXAMPLES 1.3.3. We give some examples based on the set of natural numbers.

(1) The set of squares $X = \{1, 4, 9, ..., n^2, ...\}$ is countable, since $f : \mathbb{N} \to X$ defined by $f(n) = n^2$ is bijective.
(2) The set of odd numbers $X = \{1, 3, 5, ..., 2n-1, ...\}$ is countable, since $f : \mathbb{N} \to X$ defined by $f(n) = 2n - 1$ is a bijection.

Let us state a characterization of countable sets.

LEMMA 1.5. *A set $X$ is countable.* $\Leftrightarrow$ *There exists a surjective map* $f : \mathbb{N} \to X$.

PROOF. ($\Rightarrow$) Suppose $X$ is countable. Then there is a surjection $f : \mathbb{N} \to X$ which is in addition injective.

($\Leftarrow$) Given a surjective map $f : \mathbb{N} \to X$. We have to turn this map into an bijection $g$. The idea is to omit the repeated values of $f$. We proceed in a recursive manner. Define $g(1) := f(1)$. Suppose we have chosen $n$ distinct values $g(1), g(2), ..., g(n)$. We collect the set of natural numbers where the values of $f$ are not already included among the list $\{g(1), g(2), ..., g(n)\}$:

$$X_n := \{k \in \mathbb{N} : \ f(k) \neq g(j) \text{ for every } j = 1, 2, ..., n\}.$$

The set $X_n$ can either be empty or not. Suppose $X_n = \emptyset$. Then $g : \{1, 2, ..., n\} \to X$ is a bijection and thus $X$ is finite. Otherwise, if $X_n \neq \emptyset$, then we denote by $k_n$ the least integer in $X_n$ and set $g_{n+1} := f(k_n)$. Note that by construction $g(n+1)$ differs from $g(1), g(2), ..., g(n)$. We continue in this manner. If the process terminates, then $X$ is finite, or we go through all the values of $f$ and obtain a surjection $g : \mathbb{N} \to X$. $\square$

The assignment of the number of elements of $\{0, 1, ..., n-1\}$ with the set $n$ yields that for any set $X$, there is at most one natural number $n$ such that $X$ is bijective with the set $n$.

PROPOSITION 1.3.4. *If there is a bijection between the sets $n$ and $m$, then they have the same number of elements.*

PROOF. We proceed by induction. For $n = 0$ the set $n = \{0, 1, ..., n-1\}$ is the empty set, and thus the only set bijective with it is the empty set. Suppose that $n > 0$ and that the result is true for $n - 1$. Hence there is a bijection $f : \{0, 1, ..., n-1\} \to \{0, 1, ..., m-1\}$. We assume that $f(n-1) = m-1$. Then the restriction of $f$ to the set $\{0, 1, ..., n-2\}$ gives a bijection to $\{0, 1, ..., m-2\}$. By the induction hypothesis we have $n-1 = m-1$. Let us now look at the case when $f(n-1) \neq m-1$. We have that $f(n-1) = a$ for some $a$ and $f(b) = m-1$ and we define a function $\tilde{f}$ by $\tilde{f}(x) = f(x)$ if $x \neq b, n-1$; $\tilde{f}(k) = a$ and $\tilde{f}(n-1) = m-1$. Then $\tilde{f}$ is a bijection and we conclude as before that $n = m$. $\square$

We move on to sets that are bijective to the set of natural numbers $\mathbb{N} = \{1, ...\}$.

PROPOSITION 1.3.5. *A set is at most countable it is finite or countable.*

PROOF. Suppose $f : X \to \mathbb{N}$ is an injective function. We construct a function $g : X \to \mathbb{N}$ as follows: $g(x) = n$ if $f(x)$ is the nth element in the image of f. $\square$

PROPOSITION 1.3.6. $\mathbb{N} \times \mathbb{N}$ *is countable.*

PROOF. The argument starts out with decomposing $N \times N$ into finite sets $F_0, F_1, ...,$ where

$$F_k = \{(i, j) \in \mathbb{N} \times \mathbb{N} | \, i + j = k\}$$

and the cardinality of $F_k$ is $k + 1$. Now we arrange these sets: first writing the one element of $F_0$, then the two elements of $F_1$ and so forth. Hence, we have established the assertion. In other words, we have arranged $N \times \mathbb{N}$ in a table:

$$\begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

and list the elements along sucessive (anti-)diagonals from bottom-left to top-right as

$$(1, 1), (2, 1)(1, 2), (3, 1), (2, 2), (1, 3), ....$$

We define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(n) := nth$ pair in this order. Note that $f$ is a bijection. $\square$

Here are some facts about countable sets.

Proposition 1.3.7. *We have the following assertions:*

(1) *The Cartesian product of two countable sets is countable.*
(2) *The union of countably many countable sets is countable.*

Proof. (1) We show that the Cartesian product of two countable sets is countable which reduces to the statement that the set $N \times \mathbb{N}$ is countable which we have shown in 1.3.6.
(2) Let $X_0, X_1, ...$ be a countable family of countable sets. We denote the elements of $X_i$ by $\{x_{0i}, x_{1i}, ...\}$ for $i = 0, 1, ...$ and define a map by $f(i, j) = x_{ij}$. Note that $f : N \times \mathbb{N} \to \cup_{i=0}^{\infty} X_i$ and thus the union $\cup_{i=0}^{\infty} X_i$. is countable. The map $f$ is not injective in general, because the $X_i$'s need not to be disjoint. Proposition **??** yields the desired claim.

□

Proposition 1.3.8. *The sets $\mathbb{Z}$ of integers and $\mathbb{Q}$ of rational numbers are countable.*

Proof. One of the problems of problem set 1. □

Bernstein and Schröder observed an elementary characterization of two sets having the same cardinality, we state it without proof.

Theorem 1.6. *Let $X$ and $Y$ be two sets. Suppose there are injective maps $f : X \to Y$ and $g : Y \to X$. Then there exists a bijection between $X$ and $Y$.*

We give some examples of a non-countable sets.

Theorem 1.7 (Cantor). *The set $\mathbb{R}$ of real numbers is* **not** *countable.*

If a set is not countable, then one often calls it uncountable.

Proof. We argue by contradiction and assume that $\mathbb{R}$ is countable. Then a subset of $\mathbb{R}$ is also countable. Thus the open interval $(0, 1)$ is a countable set, i.e.

$$(0, 1) = \{x_0, x_1, ...\}.$$

Any $a_i \in (0, 1)$ has an infinite decimal expansion (possibly terminating, in which case we let it continue forever with zeros):

$$a_i = 0.a_{i0}a_{i1}..., \qquad a_{ij} \in \{0, 1, ..., 9\}.$$

We set $b_i$ to be

$$b_i = \begin{cases} 3 & \text{if } a_{ii} \neq 3 \\ 1 & \text{if } a_{ii} = 7. \end{cases}$$

By construction we have $b_i \neq a_i i$ and thus the number

$$a = 0.b_1 b_2...$$

differs from $a_i$. Note that $a \in (0,1)$ which is not included in the given enumeration of $(0,1)$. Hence we have deduced a contradiction to the countability of $(0,1)$. The number $b_i \in (0,1)$ and differs from $a_i$, since the ith place of $a_i$ and $b_i$ are by construction not the same digit. □

PROPOSITION 1.3.9. *Let $X$ be the set of all binary sequences: $X = \{(a_1, a_2, a_3, ...) : a_i \in \{0,1\}\}$. Then $X$ is not countable.*

PROOF. We apply the method from the preceding theorem, aka diagonal argument.
Suppose $X = \{(x_1, x_2, x_3, ...) : x_i \in \{0,1\}\}$ is countable. Then we have

$$\begin{aligned} x_1 &= 010100....\\ x_2 &= 101111....\\ &\vdots \end{aligned}$$

Then we define a sequence $x \notin X$ by moving down the diagonal and switching the values from 0 to 1 or from 1 to 0. Hence $X$ is uncountable. □

PROPOSITION 1.3.10. *The power set $\mathcal{P}(\mathbb{N})$ of the natural numbers $\mathbb{N}$ is uncountable.*

PROOF. Let $C = \cup_{n \in \mathbb{N}}$ be a countable collection of subsets of $\mathbb{N}$. Define $X \subset \mathbb{N}$ by

$$X = \{n \in \mathbb{N} : n \in X_n\}$$

. Claim: $X \neq X_n$ for every $n \in \mathbb{N}$. Since either $n \in X$ and $n \notin X_n$ or $n \notin X$ and $n \in X_n$.
Thus $X \notin C$ and so no countable collection of subsets of $\mathbb{N}$ includes all of the subsets of $\mathbb{N}$. □

We introduce two crucial notions: the infimum and supremum of a set. First we provide some preliminaries.

DEFINITION 1.3.11. Let $A$ be a non-empty subset of $\mathbb{R}$

- If there exists $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$, then $M$ is an *upper bound* of $A$. We call $A$ *bounded above.*
- If there exists $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in A$, then $m$ is a *lower bound* of $A$.
- If there exist lower and upper bounds, then we say that $A$ is *bounded.* We call $A$ *bounded below.*

Definition 1.3.12 (Infimum and Supremum). Let $A$ be a subset of $\mathbb{R}$.

- If $m$ is a lower bound of $A$ such that $m \geq m'$ for every lower bound $m'$, then $m$ is called the *infimum* of $A$, denoted by $m = \inf A$. Furthermore, if $\inf A \in A$, then we call it the minimum of $A$, $\min A$.
- If $M$ is an upper bound of $A$ such that $M' \geq M$ for every upper bound $M'$, then $M$ is called the *supremum* of $A$, denoted by $M = \sup A$.Furthermore, if $\sup A \in A$, then we call it the maximum of $A$, $\max A$.

Note that the infimum of a set $A$, as well as the supremum, are unique. The elementary argument is left as an exercise.
If $A \subset \mathbb{R}$ is not bounded above, then we define $\sup A = \infty$. Suppose that a subset $A$ of $\mathbb{R}$ is not bounded below, then we assign $-\infty$ as its infimum.
We state a different formulation of the notions $\inf A$ and $\sup A$ that is just a reformulation of the definition.

Lemma 1.8. *Let $A$ be a subset of $\mathbb{R}$.*

- *Suppose $A$ is bounded above. Then $M \in \mathbb{R}$ is the supremum of $A$ if and only if the following two conditions are satisfied:*
  (1) *For every $a \in A$ we have $a \leq M$.*
  (2) *Given $\varepsilon > 0$, there exists $a \in A$ such that $M - \varepsilon < a$.*
- *Suppose $A$ is bounded below. Then $m \in \mathbb{R}$ is the infimum of $A$ if and only if the following two conditions are satisfied:*
  (1) *For every $a \in A$ we have $m \leq a$.*
  (2) *Given $\varepsilon > 0$, there exists $a \in A$ such that $a < m + \varepsilon$.*

Lemma 1.9. *Suppose $A$ is a bounded subset of $A$. Then* $\inf A \leq \sup A$

For $c \in \mathbb{R}$ we define the *dilate* of a set $A$ by $cA := \{b \in \mathbb{R} : b = ca \quad \text{for}\, a \in A\}$.

Lemma 1.10 (Properties). *Suppose $A$ is a subset of $\mathbb{R}$.*

(1) *For $c > 0$ we have* $\sup cA = c \sup A$ *and* $\inf cA = c \inf A$.
(2) *For $c < 0$ we have* $\sup cA = c \inf A$ *and* $\inf cA = c \sup A$.
(3) *Suppose $A$ is contained in a subset $B$. If* $\sup A$ *and* $\sup B$ *exist, then* $\sup A \leq \sup B$. *In words, making a set larger, increases its supremum.*
(4) *Suppose $A$ is contained in a subset $B$. If* $\inf A$ *and* $\inf B$ *exist, then* $\inf A \geq \inf B$. *In words, making a set smaller increases its infimum.*

(5) *Suppose* $A \subset B$ *are non-empty subsets of* $\mathbb{R}$ *such that* $x \leq y$ *for all* $x \in A$ *and* $y \in B$*. Then* $\sup A \leq \inf B$*.*
(6) *If* $A$ *and* $B$ *are non-empty subsets of* $\mathbb{R}$*, then* $\sup(A+B) = \sup A + \sup B$ *and* $\inf(A+B) = \inf A + \inf B$

PROOF. (1) We prove that $\sup cA = c \sup A$ for positive c. Suppose $c > 0$. Then $cx \leq M \Leftrightarrow x \leq M/c$. Hence $M$ is an upper bound of $cA$ if and only if $M/c$ is an upper bound of $A$. Consequently, we have the desired result.

(2) Without loss of generality we set $c = -1$. Let $a \in A$ (we assume that the set $A$ is non-empty, otherwise there is nothing interesting here). Then as a lower bound for $A$, $\inf A \leq a$. Moreover, as an upper bound for $A$, $a \leq \sup A$. Using transitivity, we conclude that $\inf A \leq \sup A$.

We now prove the second identity. Keep in mind that the supremum of a set is its **least upper bound**, while the infimum is its **greatest lower bound**.

For any $a \in A$, $\inf A \leq a$, so $-\inf A \geq -a$, showing that $-\inf A$ is an upper bound for $-A$. Therefore, $-\inf A \geq \sup(-A)$, which implies $\boxed{\inf A \leq -\sup(-A)}$.

For any $a \in A$ we have $-a \in -A$, so $-a \leq \sup(-A)$, which implies $a \geq -\sup(-A)$. Therefore, $-\sup(-A)$ is a lower bound for $A$, so $\boxed{-\sup(-A) \leq \inf A}$.

The two boxed inequalities prove the identity $\inf A = -\sup(-A)$.

(3) Since $\sup B$ is an upper bound of $B$, it is also an upper bound of $A$, i.e. $\sup A \leq \sup B$.

(4) Analogously to (iii).

(5) Since $x \leq y$ for all $x \in A$ and $y \in B$, $y$ is an upper bound of $A$. Hence $\sup A$ is a lower bound of $B$ and we have $\sup A \leq \inf B$.

(6) By definition $A + B = \{c : c = a + b \text{ for some } a \in A, b \in B\}$ and thus $A + B$ is bounded above if and only if $A$ and $B$ are bounded above. Hence $\sup(A+B) < \infty$ if and only if $\sup A$ and $\sup B$ are finite. Take $a \in A$ and $b \in B$, then $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound of $A + B$:

$$\sup(A+B) \leq \sup A + \sup B.$$

The reverse direction is a little bit more involved. Let $\varepsilon > 0$. Then there exists $a \in A$ and $b \in B$ such that

$$a > \sup A - \varepsilon/2, \quad b > \sup B - \varepsilon/2.$$

Thus we have $a + b > \sup A + \sup B - \varepsilon$ for every $\varepsilon > 0$, i.e. $\sup(A + B) \geq \sup A + \sup B$.

The other statements are assigned as exercises. $\square$

One reason for the relevance of the notions of supremum and infimum is in the formulation of properties of functions.

Definition 1.3.13. Let $f$ be a function with domain $X$ and range $Y \subseteq \mathbb{R}$. Then

$$\sup_X f = \sup\{f(x) : x \in X\}, \quad \inf_X f = \inf\{f(x) : x \in X\}.$$

If $\sup_X f$ is finite, then $f$ is bounded from above on $A$, and if $\inf_X f$ is finite we call $f$ bounded from below. A function is bounded if both the supremum and infimum are finite.

Lemma 1.11. *Suppose that $f, g : X \to \mathbb{R}$ and $f \leq g$, i.e. $f(x) \leq g(x)$ for all $x \in X$. If $g$ is bounded from above, then $\sup_X f \leq \sup_A g$. Assume that $f$ is bounded from below. Then $\inf_X f \leq \inf_X g$.*

Proof. Follows from the definitions. $\square$

The supremum and infimum of functions do not preserve strict inequalities. Define $f, g : [0, 1] \to \mathbb{R}$ by $f(x) = x$ and $g(x) = x + 1$. Then we have $f < g$ and

$$\sup_{[0,1]} f = 1, \quad \inf_{[0,1]} f = 0, \quad \sup_{[0,1]} g = 2, \quad \inf_{[0,1]} g = 1.$$

Hence we have $\sup_{[0,1]} f > \inf_{[0,1]} g$.

Lemma 1.12. *Suppose $f, g$ are bounded functions from $X$ to $\mathbb{R}$ and $c$ a positive constant. Then*

$$\sup_X (f + cg) \leq \sup_X f + c \sup_X g \quad \inf_X (f + cg) \geq \inf_X f + c \inf_X g.$$

The proof is left as an exercise. Try to convice yourself that the inequalities are in general strict, since the functions $f$ and $g$ may take values close to their suprema/infima at different points in $X$.

Lemma 1.13. *Suppose $f, g$ are bounded functions from $X$ to $\mathbb{R}$. Then*

$$|\sup_X f - \sup_X g| \leq \sup_X |f - g|, \quad |\inf_X f - \inf_X g| \leq \sup_X |f - g|$$

Lemma 1.14. *Suppose $f, g$ are bounded functions from $X$ to $\mathbb{R}$ such that*

$$|f(x) - f(y)| \leq |g(x) - g(y)| \quad \textit{for all } x, y \in X.$$

*Then*

$$\sup_X f - \inf_X f \leq \sup_X g - \inf_X g.$$

Recall that a sequence $(x_n)$ of real numbers is an ordered list of numbers $x_n$, indexed by the natural numbers. In other words, $(x_n)$ is a function $f$ from $\mathbb{N}$ to $\mathbb{R}$ with $f(n) = x_n$. A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$, so the properties of the inf and sup for functions apply to sequences as well.

CHAPTER 2

# Normed spaces and innerproduct spaces

In order to measure the length of a vector and to define a distance between vectors we introduce the notion of a norm of a vector. Norms may be a tool to specify properties of a class of vectors in a convenient form. We review basic aspects of vector spaces before we define normed vector spaces.

## 2.1. Vector spaces

Vector spaces and linear mappings between them are a useful tool for engineers, scientists and mathematicians, aka Linear Algebra.

Vector spaces formalize the notion of linear combinations of objects that might be vectors in the plane, polynomials, smooth functions, sequences. Many problems in engineering, mathematics and science are naturally formulated and solved in this setting due to their linear nature. Vector spaces are ubiquitous for several reasons, e.g. as linear approximation of a non-linear object, or as building blocks for more complicated notions, such as vector bundles over topological spaces. We restrict our discussion to complex and real vector spaces.

A set V is a vector space if it is possible to build linear combinations out of the elements in V. More formally, on $V$ we have the operations of addition of vectors and multiplication by scalars. The scalars will be taken from a field $\mathbb{F}$, which is either the real numbers $\mathbb{R}$ or $\mathbb{C}$. In various situations $\mathbb{F}$ might also be a finite field or a field different from $\mathbb{R}$ and $\mathbb{C}$. If it is necessary we will refer to these vector spaces as real or complex vector spaces.

Developing an understanding of these vector spaces is one of the main objectives of this course. The axioms for a vector space specify the properties that addition of vectors and scalar multiplication.

Definition 2.1.1. A *vector space* over a field $\mathbb{F}$ is a set $V$ together with the operations of addition $V \times V \to V$ and scalar multiplication $\mathbb{F} \times V \to V$ satisfying the following properties:

(1) Commutativity: $u + v = v + u$ for all $u, v \in V$ and $(\lambda\mu v) = \lambda(\mu v)$ for all $\lambda, \mu \in \mathbb{F}$;
(2) Associativity: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$;
(3) Additive identity: There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$ ;
(4) Additive inverse: For every $v \in V$ , there exists an element $w \in V$ such that $v + w = 0$;
(5) Multiplicative identity: $1v = v$ for all $v \in V$ ;
(6) Distributivity: $\lambda(u + v) = \lambda u + \lambda v$ and $(\lambda + \mu)u = \lambda u + \mu u$ for all $u, v \in V$ and $\lambda, \mu \in \mathbb{F}$.

The elements of a vector space are called vectors. Given $v_1, ..., v_n$ be in $V$ and $\lambda_1, ..., \lambda_n \in \mathbb{F}$ we call the vector

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

a *linear combination.*

Our focus will be on three classes of examples.

Examples 2.1.2. We define some useful vector spaces.

- **Spaces of $n$-tuples:** The set of tuples $(x_1, ..., x_n)$ of real and complex numbers are vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with respect to component-wise addition and scalar multiplication: $(x_1, ..., x_n)+(y_1, ..., y_n) = (x_1+y_1, ..., x_n+y_n)$ and $\lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n)$.
- The set of functions $\mathbb{F}(X, Y)$ of a set $X$ to a set $Y$: $\lambda f + \mu g(x) := (\lambda f + \mu g)(x)$ for all $x \in X$.
- The space of polynomials of degree at most $n$, denoted by $\mathcal{P}_n$, where we define the operations of multiplication and addition coefficient-wise: For $p(x) = a_0 + a_1 x + \cdots a_n x^n$ and $q(x) = b_0 + b_1 x + \cdots b_n x^n$ we define

$$(p+q)(x) = (a_0+b_0)+(a_1+b_1)x+\cdots(a_n+b_n)x^n \text{ and } (\lambda p)(x) = \lambda a_0+\lambda a_1 x+\cdots \lambda a_n x^n$$

  for $\lambda \in \mathbb{F}$.

  The space of all polynomials $\mathcal{P}$ is the vector space of polynomials of arbitrary degrees.
- **Sequence spaces**: $s$ denotes the set of sequences, $c$ the set of all convergent sequences, $c_0$ the set of all convergent sequences tending to 0, $c_f$ the set of all sequences with finitely many non-zero elements.
- **Function spaces:** The set of continuous functions $C(I)$ on an interval of $\mathbb{R}$, popular choices for $I$ are $[0, 1]$ and $\mathbb{R}$. We define

addition and scalar multiplication as follows: For $f, g \in C(I)$ and $\lambda \in \mathbb{F}$

$$(f+g)(x) = f(x) + g(x) \quad \text{and}\, (\lambda f)(x) = \lambda f(x).$$

We denote by $C^{(n)}(I)$ the space of $n$-times continuously differentiable functions on $I$ and the space $C^\infty(I)$ of smooth functions on $I$ is the space of functions with infinitely many continuous derivatives. More generally, the set $\mathcal{F}(X)$ of functions from a set $X$ to $\mathbb{F}$ is a vector space for the operations defined above. Note that $\mathcal{F}(\{1, 2, ..., n\})$ is just $\mathbb{F}^n$ and hence the first class of examples.

- **Spaces of matrices**: Denote by $\mathcal{M}_{m\times n}(\mathbb{C})$ the space of complex $m \times n$ matrices where we define addition and scalar multiplication entry-wise: For $A = (a_{ij})_{i,j}$ and $B = (a_{ij})_{i,j}$ where $i = 1, ..., m$ and $j = 1, ...n$ we define

$$A + B := (a_{ij} + b_{ij})_{i,j} \qquad \text{and}\; \alpha(a_{ij})_{ij} = (\alpha a_{ij})_{ij},\ \alpha \in \mathbb{F}.$$

There are relations between the vector spaces in the aforementioned list. We start with clarifying their inclusion properties.

Definition 2.1.3. A subset $W$ of a vector space $V$ is called a *subspace* if $W$ is a vector subspace with respect to addition and scalar multiplication of $V$.

One way to express this more concretely is stated in the next lemma:

Lemma 2.1. *A subset $W$ of a vector space $V$ is a subspace if and only if $W$ is closed under linear combinations: For any $\alpha, \beta \in \mathbb{F}$ and $w_1, w_2 \in W$ we have $\alpha_1 w_1 + \alpha_2 w2 \in W$. Equivalently, we have that the subset $W$ of a vector space $V$ is a subspace if and only if*

(1) $0 \in W$;
(2) $w_1 + w_2 \in W$ *for any* $w_1, w_2 \in W$;
(3) $\alpha w$ *for any* $\alpha \in \mathbb{F}$ *and any* $w \in W$.

Consequently, we have a way to decide when a subset of a vector space is not a subspace.

Lemma 2.2. *A subset $W$ of a vector space $V$ is a not a subspace if one of the following conditions holds:*

(1) $0 \notin W$;
(2) *There are some* $w_1, w_2 \in W$ *such that* $w_1 + w_2 \in W$ ;
(3) *There is a vector* $w \in W$ *such that* $-w$ *is not in* $= W$.

This is the contrapositve of the preceding lemma.

Here are some examples of vector subspaces:

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{F}, \qquad C^\infty(I) \subset C^{(n)}(I) \subset C(I), \qquad c_f \subset c_0 \subset c \subset s$$

.

We define the linear span, span$S$, of a subset $S$ of a vector space $V$ to be the intersection of all subspaces of $V$ containing $S$.

Linear transformations $T$ between vector spaces $V$ and $W$ are mappings $T$ that respect linear transformations:

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) \qquad \text{for any } v_1, v_2 \in V, \alpha, \beta \in \mathbb{F}.$$

We denote by $\mathcal{L}(V, W)$ the set of all linear transformations between $V$ and $W$ and it is a subset of the vector space of all functions $f : V \to W$. Furthermore $\mathcal{L}(V, W)$ is a vector space:

$$\mathcal{L}(V, W) \subseteq \mathcal{F}(V, W).$$

EXAMPLE 2.1.4. Let $D$ denote the differentiation operator $Df = f'$. Then $D : C^{(1)}(a, b) \to C(a, b)$ is a linear transformation.

Linear transformations have some useful properties.

LEMMA 2.3. *For any $T \in \mathcal{L}(V, W)$ we have $T(0) = 0$.*

PROOF. We have that $v + 0 = v$ for any $v \in V$; in particular for $v = 0$:

$$T(0) = T(0 + 0) = T(0) + T(0)$$

and after subtracting $T(0)$ we get $T(0) = 0$. □

The *kernel* of $T \in \mathcal{L}(V, W)$ is the set

$$\ker(T) := \{v \in V \,|\, Tv = 0\},$$

i.e. $\ker(T) = T^{-1}(0)$.

LEMMA 2.4. *For a linear transformation $T : V \to W$ the kernel of $T$ is a subspace of $V$.*

PROOF. Suppose $v_1, v_2 \in \ker(T)$. Then for any scalars $\alpha_1, \alpha_2$ we have

$$T(\alpha v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$$

and thus $\alpha v_1 + \alpha_2 v_2 \in \ker(T)$. □

The range of $T$ is a subspace of $W$, too.

LEMMA 2.5. *The range of a linear transformation $T : V \to W$ is a subspace of $W$.*

Proof. Exercise, see problem set 2. □

There is some natural operations for vector spaces.

Definition 2.1.5. Let $V$ and $W$ be subspaces of $Z$.

(1) The **sum** of $V$ and $W$ is defined by $V + W := \{z \in Z|\, z = v + w\ v \in V, w \in W\}$.
(2) The **intersection** of $V$ and $W$ is defined by $V \cap W := \{z \in Z|\, z \in V \cap W\}$.

From the definitions we see that $V + W$ and $V \cap W$ are subspace of $Z$. We introduce some more notions: If the sum of the subspaces $V$ and $W$ equals $Z$, then we say that $Z$ is the sum of $V$ and $W$, i.e. $V + W = Z$. If in addition, the subspaces are disjoint subsets, $U \cap V = \{0\}$, then we refer to the sum of $V$ and $W$ as the **direct sum**.

Lemma 2.6. *Let $I$ be an index set. Given vector spaces $V_i$ for any $i \in I$. Then $\cap_{i \in I} V_i$ is a vector space.*

Proof. Exercise, see problem set 2. □

Definition 2.1.6. Let $S$ be a nonempty subset of a vector space $V$. Then we define the **span** of $S$, $\mathrm{span}(S)$, as the intersection of all subspaces of $V$ that contain $S$.

Lemma 2.7. *Let $S \subset V$ be a nonempty subset. Then*

$$\mathrm{span}(S) = \{\lambda_1 v_1 + \ldots + \lambda_n v_n \ :\ v_1, \ldots, v_n \in S \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{F}\}.$$

By definition, $\mathrm{span}(S)$ is the intersection of all subspaces $W$ of $V$ that contain the set $S$. From the preceding lemma, it follows that $\mathrm{span}(S)$ is a subspace of $V$, hence it is the *smallest* subspace of $V$ that contains $S$.

Let us denote

$$W := \{\lambda_1 v_1 + \ldots + \lambda_n v_n \ :\ v_1, \ldots, v_n \in S \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{F}\},$$

so $W$ is the set of all linear combinations with elements in $S$.

Being a subspace of $V$, $\mathrm{span}(S)$ must contain all such linear combinations, so we must have that

$$W \subset \mathrm{span}(S).$$

All we have left to show is that $W$ is a subspace of $V$. This is not hard to see, since linear combinations of linear combinations are linear combinations as well.

Indeed, let $a, b \in \mathbb{F}$ and let $w_1, w_2 \in W$, so

$$\begin{aligned} w_1 &= \lambda_1 v_1 + \ldots + \lambda_n v_n \quad \text{with } v_1, \ldots, v_n \in S, \\ w_2 &= \mu_1 u_1 + \ldots + \mu_m u_m \quad \text{with } u_1, \ldots, u_m \in S. \end{aligned}$$

Then

$$a\,w_1 + b\,w_2 = a\lambda_1\,v_1 + \ldots + a\lambda_n\,v_n + b\mu_1\,u_1 + \ldots + b\mu_m\,u_m,$$

and since $v_1, \ldots, v_n, u_1, \ldots, u_m \in S$, it follows that $a\,w_1 + b\,w_2 \in W$.

Therefore, $W$ is a subspace of $V$ that contains $S$, so we must have

$$\operatorname{span}(S) \subset W.$$

Together with the previous inclusion, this proves the equality of the two sets.

## 2.2. Normed spaces

The norm on a general vector space generalizes the notion of the length of a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 2.2.1. A *normed space* is a vector space $X$ together with a function $\|.\| : X \to \mathbb{R}$, the *norm* on $X$, such that for all $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) *Positivity:* $0 \leq \|x\| < \infty$ and $\|x\| = 0$ if and only if $x = 0$;
(2) *Homogeneity:* $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{F}$;
(3) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$.

We denote this normed space by $(X, \|.\|)$

A norm gives a way to measure the distance between two vectors by $d(x, y) := \|x - y\|$. We refer to $d$ as the metric associated to the norm $\|.\|$.

Proposition 2.2.2. *Let $(X, \|.\|)$ be a normed space. Then $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ satisfies for all $x, y, z \in X$*

(i) *$d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$ (positivity);*
(ii) *$d(x, y) = d(y, x)$ (symmetry);*
(iii) *$d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).*

Proof. The properties (i)-(iii) are direct consequences of the axioms for a norm. In particular, (i) follows from property (1) of a norm, (ii) is derived from property (ii) of a norm for $\lambda = -1$ and (iii) is deduced from property (3) of a norm. □

The metric $d$ on $X$ is also compatible with the linear structure of a vector space:

- *Translation invariance:* $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$;
- *Symmetry:* $d(\alpha x, \alpha y) = |\alpha| d(y, x)$ for all $x, y \in X$ and $\alpha \in \mathbb{F}$.

The function $d(x, y) = \|x-y\|$ on the vector space $\mathbb{R}$ is an example of a distance function on $\mathbb{R}$, aka as a metric.

The metric $d$ on $X$ gives us a way to generalize intervals in $\mathbb{R}$ to so-called balls.

Definition 2.2.3. For $r > 0$ and $x \in X$ we define the *open ball* $B_r(x)$ of radius $r$ and center $x$ as the set

$$B_r(x) = \{y \in X : \|x - y\| < r\},$$

and the *closed ball* $\overline{B}_r(x)$ of radius $r$ and center $x$ as

$$\overline{B}_r(x) = \{y \in X : \|x - y\| \leq r\}.$$

The translation invariance and the homogeneity imply that the ball $B_r(x)$ is the image of the unit ball $B_1(0)$ centered at the origin under the (affine) mapping $f(y) = ry + x$.

The balls $B_r(x)$ have another peculiar feature. Namely, these are convex subsets of $X$.

Definition 2.2.4. Let $X$ be a vector space.

- For two points $x, y \in X$ the *interval* $[x, y]$ is the set of points $\{z|\, z = \lambda x + (1-\lambda)y\ 0 \leq \lambda \leq 1\}$.
- A subset $E$ of $X$ is called *convex* if for any two points $x, y \in E$ the interval $[x, y]$ is also in $E$.

The notion of convexity is central to the theory of vector spaces and enters in an intricate manner in functional analysis, numerical analysis, optimization, etc. .

Lemma 2.8. *Let $(X, \|.\|)$ be a normed vector space. Then the unit ball $B_1(0) = \{x \in X|\, \|x\| \leq 1\}$ is a convex set.*

Proof. For $x, y \in B_1(0)$ we have that $\|\lambda x + (1-\lambda)y\| \leq |\lambda|\|x\| + |1-\lambda|\|y\| = 1$, because $\|x\|, \|y\|$ are both less than or equal to 1. Thus $\lambda x + (1-\lambda)y \in B_1(0)$. □

The real numbers with the absolute value is a normed space $(\mathbb{R}, |.|)$ and the open ball $B_r(x)$ is the open interval $(x-r, x+r)$ and $\overline{B}_r(x)$ is the closed interval $[x-r, x+r]$.

Lemma 2.9 (Reverse triangle inequality). *Let $(X, \|.\|)$ be a normed space. Then we have*

$$|\|x\| - \|y\|| \leq \|x - y\| \qquad \text{for all } x, y \in X.$$

Proof. See problem set 3. □

A fundamental class of normed spaces is $\mathbb{R}^n$ with the $\ell^p$-norms.

DEFINITION 2.2.5. For $p \in [1, \infty)$ we define the **p-norm**, denoted by $\|.\|_p$, on $\mathbb{R}^n$ by assigning to $x = (x_1, ..., x_n) \in \mathbb{R}^n$ the number $\|x\|_p$:

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots |x_n|^p)^{1/p}$$

. For $p = \infty$ we define the $\ell^\infty$-norm $\|.\|_\infty$ on $\mathbb{R}$ by

$$\|x\|_\infty = \max |x_1|, ..., |x_n|.$$

The notation for $\|.\|_\infty$ is justified by the fact that it is the limit of the $\|.\|_p$-norms.

LEMMA 2.10. *For $x \in \mathbb{R}^n$ we have that*

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p.$$

PROOF. Without loss of generality we assume that the largest component of $x$, the $\|x\|_\infty$, to be $x_n$. For $1 \leq p < \infty$ we have

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots |x_n|^p)^{1/p} = \|x\|_\infty ((\frac{|x_1|}{\|x\|})^p + \frac{|x_2|}{\|x\|})^p + \cdots + 1)^{1/p},$$

since $\frac{|x_i|}{\|x\|}) < 1$ for $i = 1, ..., n-1$ we have $\lim_{p \to \infty} \frac{|x_1|}{\|x\|})^p = 0$. Thus we have

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty.$$

□

In the proof of the triangle inequality for the p-norms we have to rely on some inequalities: Hölder's inequality and Young's inequality.

For $p \in (1, \infty)$ we define its *conjugate* $q$ as the number such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $p = 1$, then we define its conjugate $q$ to be $\infty$ and vice versa for $p = \infty$ we set $q = 1$.

LEMMA 2.11 (Young's inequality). *For $p \in (1, \infty)$ and $q$ its conjugate we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

*for any non-negative real numbers $a, b$.*

Proof. Consider the function $f(x) = x^{p-1}$ and integrate this with respect to $x$ from zero to $a$. Now take the inverse function of $f$ given by $f^{-1}(y) = y^{q-1}$, where we used that $1/(p-1) = q-1$ for conjugate exponents $p$ and $q$. Let us integrate $f^{-1}$ from zero to $b$. Then the sum of these two integrals always exceeds the product $ab$, see figure. Note that the two integrals are given by $a^p/p$ and $b^q/q$. Hence we have established Young's inequality. □

A consequence of Young's inequality is Hölder's inequality.

Lemma 2.12. *Suppose $p \in (1,\infty)$ and $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are vectors in $\mathbb{R}^n$. Then*

$$|\sum_{i=1}^n x_i y_i \le \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}\Big(\sum_{i=1}^n |y_i|^q\Big)^{1/q}.$$

Proof. Set $a_i = |x_i|/(\sum_{i=1}^n |x_i|^p)^{1/p}$ and $b_i = |y_i|/(\sum_{i=1}^n |y_i|^q)^{1/q}$. Then we have $\sum_i a_i^p = 1$ and $\sum_i b_i^q = 1$. By Young's inequality

$$\sum_{i=1}^n |x_i||y_i| \le (\sum_{i=1}^n |x_i|^p)^{1/p}(\sum_{i=1}^n |y_i|^q)^{1/q}.$$

□

The unit balls of $(\mathbb{R}^2, \|.\|_1)$, $(\mathbb{R}^2, \|.\|_2)$ and $(\mathbb{R}^2, \|.\|_\infty)$ indicate the different nature of these norms.

Proof. Positivity and homogeneity are consequences of the corresponding properties of the absolute value of a real number. The triangle inequality is the non-trivial assertion that we split up in three cases $p = 1$, $p = \infty$ and $p \in (1,\infty)$. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be points in $\mathbb{R}^n$.

(1) For $p = 1$ we have

$$\|x+y\|_1 = |x_1+y_1|+\cdots+|x_n+y_n| \le |x_1|+|y_1|+\cdots+|x_n|+|y_n| \le \|x\|_1+\|y\|_1$$

.

(2) For $p = \infty$ the argument is similar:

$$\begin{aligned}\|x+y\|_\infty &= \max\{|x_1+y_1|, ..., |x_n+y_n|\}\\ &\le \max\{|x_1|+|y_1|, ..., |x_n|+|y_n|\}\\ &\le \max\{|x_1|, ..., |x_n|\} + \max\{|y_1|, ..., |y_n|\} = \|x\|_\infty + \|y\|_\infty.\end{aligned}$$

(3) The general case $p \in (1, \infty)$: The triangle inequality follows from Hölder's inequality.

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\
&\leq \sum_{i=1}^n |x_i + y_i|^{p-1}(|x_i| + |y_i|) \\
&\leq \sum_{i=1}^n |x_i + y_i|)^{p-1}|x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1}|y_i| \\
&\leq \Big(\sum_{i=1}^n |x_i + y_i|^p\Big)^{1/q}\Big(\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^n |y_i|^p\Big)^{1/p}\Big) \\
&= \|x+y\|_p^{1/q}(\|x\|_p + \|y\|_p)
\end{aligned}$$

Dividing by $\|x+y\|_p^{1/q}$ and using $1 - 1/q = 1/p$ we obtain the triangle inequality:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Thus the space $\mathbb{R}^n$ with the $p$-norm $\|.\|_p$ is a normed space for $p \in [1, \infty]$.

□

The triangle inequality for p-norms on $\mathbb{R}^n$ is also known as **Minkowski's inequality**:

$$(\sum_{i=1}^n |x_i + y_i|^p)^{1/p} \leq (\sum_{i=1}^n |x_i|^p)^{1/p} + (\sum_{i=1}^n |y_i|^p)^{1/p}.$$

There are variations of the $(\mathbb{R}^n, \|.\|_p)$ with relevance in engineering, physics and mathematics. (i) Replace the real scalars by complex scalars $(\mathbb{C}^n, \|.\|_p)$; (ii) Replace $\mathbb{R}^n$ by the vector space of sequences $s$; (iii) Deal with complex-valued sequences, (iv) Consider continuous functions and define norms in terms of integrals instead of sums for sequences.

Before we present these classes of normed spaces, we show that the vector space of $m \times n$-matrices is a normed spaces, too.

Define a norm on $\mathcal{M}_{m\times n}(\mathbb{F})$ by picking a norm on $\mathbb{F}^{mn}$: For $1 \leq p < \infty$ we define $\|A\|_{(p)} = (\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p)^{1/p}$ or $\|A\|_{(\infty)} = \max |a_{ij}|$ for $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The case $p = 2$ is of special interest and is known as the Frobenius norm.

PROPOSITION 2.2.6. *For $1 \leq p \leq \infty$ we have that $(\mathcal{M}_{m\times n}(\mathbb{F}), \|.\|_p)$ is a normed space.*

The identification of $\mathcal{M}_{m\times n}(\mathbb{F})$ with the vector space $\mathbb{F}^{mn}$ gives us this result.

PROPOSITION 2.2.7. *Let $\mathbb{C}^n$ be the vector space of complex n-tuples $z = (z_1, ..., z_n)^T$, $z_i \in \mathbb{C}$ for $i = 1, ..., n$. For $1 \leq p < \infty$ we define*

$$\|z\|_p = (\sum_{i=1}^{n})|z_i|^p)^{1/p}, \ z \in \mathbb{C}^n$$

*and for $p = \infty$ we have $\|z\|_\infty := \max|z_i| : i = 1, ..., n$. where $z_i \in \mathbb{C}$ and $|z_i| = (z_i\overline{z_i})^{1/2}$ denotes the modulus of $z_i$. Then $(\mathbb{C}^n, \|.\|)_p$ is a normed space for $1 \leq p \leq \infty$. The proof of $\mathbb{R}^n$ goes through without any changes.*

PROOF. Young's inequality is a statement about non-negative numbers which in this case are modulus of complex numbers. Hence Young's inequality is valid in this case as well and consequently Hölder's inequality. The later is the key to prove the triangle inequality. □

Recall that $s$ denotes the vector space of all sequences with values in $\mathbb{R}$ or $\mathbb{C}$. We define for $1 \leq p < \infty$ the space $\ell^p$ as the set of all sequences $x = (x_1, x_2, ...)$ satisfying

$$\|x\|_p := (|x_1|^p + |x_2|^p + \cdots)^{1/p} < \infty,$$

and $\ell^\infty$ denotes the space of all bounded sequences $(s, \|.\|_\infty)$ with

$$\|x\|_\infty := \sup_{i\in\mathbb{N}} |x_i|,$$

where $|.|$ denotes the absolute value of a real number or the modulus of a complex number, respectively.

LEMMA 2.13 (Hölder's inequality). *For $1 \leq p \leq \infty$ and $q$ its conjugate index we have for $x \in \ell^p$ and $y \in \ell^q$*

$$\sum_{i=1}^{\infty} |x_i||y_i| \leq (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}(\sum_{i=1}^{\infty} |y_i|^q)^{1/q}.$$

Since Hölder's inequality is true for all $n \in \mathbb{N}$ we deduce that the limits of the partial sums in question also satisfy these inequalities. Hence we deduce the desired inequality for sequences instead of n-tuples.

PROPOSITION 2.2.8. *For $1 \leq p \leq \infty$ we have that $\ell^p$ is a normed vector space.*

PROOF. First we show that $\ell^p$ is a vector space for $p \in [1, \infty)$: For $\alpha \in \mathbb{F}$ and $x \in \ell^p$ we have $\alpha x \in \ell^p$. One has to work a little bit to see that for $x, y \in \ell^p$ also $x + y \in \ell^p$:

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^{\infty} |x_i + y_i|^p \\
&\leq 2^p \sum_{i=1}^{\infty} \max\{|x_i|, |y_i|\}^p \\
&= 2^p \sum_{i=1}^{\infty} |\max\{|x_i|, |y_i|\}|^p \\
&\leq 2^p (\sum_{i=1}^{\infty} |x_i|^p + \sum_{i=1}^{\infty} |y_i|^p) = 2^p(\|x\|_p^p + \|y\|_p^p) < \infty.
\end{aligned}$$

The norm properties may be deduced as in the case of $\mathbb{F}^n$ since we have Hölder's inequality at our disposal. □

For $1 \leq p < \infty$ the spaces $(\ell^p, \|.\|_p)$ are subspaces of the vector space of sequences converging to zero, $c_0$. In contrast $(\ell^\infty, \|.\|_\infty)$ is the space of bounded sequences and is much larger than the other $\ell^p$-spaces. We have the following inclusions:

LEMMA 2.14. *For $p_1 < p_2$ the space $\ell^{p_1}$ is a proper subspace of $\ell^{p_2}$, i.e.*

$$\ell^1 \subset \ell^2 \subset \ell^\infty.$$

PROOF. See problem set 4. □

For example $(1/n)_n$ is in $\ell^p$ for $p \geq 2$, but not in $\ell^1$.

We finish this section with normed spaces based on continuous functions.

DEFINITION 2.2.9. For $f \in C[a, b]$ we define its p-norm for $1 \leq p < \infty$ by

$$\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$$

and $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$. We denote by $(C[a, b], \|.\|_p)$ the set of all functions satisfying $\|f\|_p < \infty$.

LEMMA 2.15 (Hölder's inequality). *For $1 \leq p \leq \infty$ and its conjugate exponent $q$ we have*

$$\int_a^b |f(x)||g(x)| dx \leq \|f\|_p \|g\|_q.$$

PROOF. We assume without loss of generality that $\|f\|_p = 1 = \|g\|_q$. By Young's inequality we have

$$|f(x)||g(x)| \le |f(x)|^p/p + |g(x)|^q/q$$

and thus

$$\int_a^b |f(x)||g(x)| \le \frac{1}{p}\int_a^b |f(x)|^p dx + \frac{1}{q}\int_a^n |g(x)|^q dx = \|f\|_p\|g\|_q.$$

As in the case of $\mathbb{F}^n$ we are able to turn this inequality in the desired one. $\square$

PROPOSITION 2.2.10. *The space* $(C[a,b], \|.\|_p)$ *is a normed space for* $p \in [1,\infty]$.

PROOF. As for $\ell^p$-spaces we deduce that the $\|.\|_p$ is a vector space. The norm part is based on the validity of Hölder's inequality as above. $\square$

We close with a way to construct a normed space out of given normed spaces. Let $\{X_1, \|.\|_{X_1}), ..., (X_1, \|.\|_{X_1})\Big\}$ be given normed spaces. Then the direct product $X_1 \times \cdots \times X_n$ is a normed space for

$$\|(x_1, ..., x_n)\| := \|x_1\|_{X_1} + \cdots + \|x_n\|_{X_n}.$$

## 2.3. Innerproduct spaces

In this section we consider innerproduct spaces and we start with the case of real vector spaces and afterwards treat complex vector spaces.

For vectors in $\mathbb{R}^3$ we have the 'dot product' aka 'scalar product' that assigns to a pair of vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ the number

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

Pythagoras' theorem gives the length of $x = (x_1, x_2, x_3)$ as $\sqrt{x_1^2 + x_2^2 + x_3^2}$. Note that $\langle x, x\rangle = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Innerproduct spaces are a generalization of these basic facts from Euclidean geometry to general vector spaces.

DEFINITION 2.3.1. Let $X$ be a real vector space. An *innerproduct* on $X$ is a map $\langle ., .\rangle : X \times X \to \mathbb{R}$ satisfying:

(1) (Linearity) For vectors $x_1, x_2, y \in X$ and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ we have $\langle \alpha_1x_1 + \alpha_2x_2, y\rangle = \alpha_1 \langle x_1, y\rangle + \alpha_2 \langle x_2, y\rangle$.
(2) (Symmetry) For vectors $x, y \in X$ we have $\langle x, y\rangle = \langle y, x\rangle$.
(3) (Positive definiteness) For any $x \in X$ we have $\langle x, x\rangle \ge 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

We call $(X, \langle ., . \rangle)$ an *innerproduct space* and define by $\|x\| := \langle x, x \rangle^{1/2}$.

Here is a reformulation of the positive definiteness of innerproducts.

LEMMA 2.16. *Suppose $X$ is an innerproduct space. If $\langle x, y \rangle = 0$ for all $y \in X$, then $x = 0$.*

PROOF. Since $\langle x, y \rangle = 0$ holds for all $y \in X$, in particular for $y = x$ and thus $\langle x, x \rangle = 0$. Hence $x = 0$. $\square$

Note that the symmetry and linearity in the first entry gives that $\langle ., . \rangle$ is bilinear: For vectors $x, y_1, y_2 \in X$ and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ we have $\langle x, \alpha_1 y_1 + \alpha_2 y_2 \rangle y = \alpha_1 \langle x, y_1 \rangle + \alpha_2 \langle x, y_2 \rangle$.

EXAMPLE 2.3.2. The family of p-norms on $\mathbb{R}^n$, the space of sequences $s$ and on the space of continuous functions $C[a, b]$ include for $p = 2$ important examples of innerproduct spaces.

There is a link between innerproducts and the length of $x$. Namely $\langle x, x \rangle^{1/2}$ is the length $\|x\|$ of $x$. The proof of this fact is based on a well-known inequality.

PROPOSITION 2.3.3 (Cauchy-Schwarz). *Suppose $X$ is a real innerproduct space. Then for all $x, y \in X$ we have*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

*We have $|\langle x, y \rangle| = \|x\| \|y\|$ if and only if $x = \alpha y$ for some $\alpha \in \mathbb{R}$.*

PROOF. For any $t \in \mathbb{R}$ and $x, y \in X$ we have $\|x - ty\| \geq 0$. More explicitly, we have

$$\begin{aligned}
\|x - ty\| &= \langle x - ty, x - ty \rangle = \langle x, x \rangle - t(\langle y, x \rangle + \langle x, y \rangle) + t^2 \langle y, y \rangle \\
&= \langle x, x \rangle - 2t \langle x, y \rangle + t^2 \langle y, y \rangle
\end{aligned}$$

Suppose $y \neq 0$, otherwise there is nothing to show.

Hence we have

$$\begin{aligned}
t^2 \langle y, y \rangle - 2t \langle x, y \rangle + \langle x, x \rangle &= \langle y, y \rangle \left( t^2 - 2t \frac{\langle x, y \rangle}{\langle y, y \rangle} + \frac{\langle x, x \rangle}{\langle y, y \rangle} \right) \\
&= \langle y, y \rangle \left( \left( t - \frac{\langle x, y \rangle}{\langle y, y \rangle} \right)^2 - \frac{\langle x, y \rangle^2}{\langle y, y \rangle^2} + \frac{\langle x, x \rangle}{\langle y, y \rangle} \right) \\
&= \langle y, y \rangle \left( \left( t - \frac{\langle x, y \rangle}{\langle y, y \rangle} \right)^2 + \frac{\langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2}{\langle y, y \rangle^2} \right)
\end{aligned}$$

Hence we have $\langle x,x\rangle \langle y,y\rangle - \langle x,y\rangle^2 \geq 0$, i.e.

$$|\langle x,y\rangle| \leq \langle x,x\rangle^{1/2} \langle y,y\rangle^{1/2}.$$

The assertion about the equality follows from the proof of the Cauchy-Schwarz inequality, since $\|x - ty\| = 0$ if and only if $x = \alpha y$ for some $\alpha \in \mathbb{R}$. □

As a consequence we deduce that innerproduct spaces $(X, \langle .,.\rangle)$ are normed spaces for $\|x\| = \langle x,x\rangle^{1/2}$.

PROPOSITION 2.3.4. *For $(X, \langle .,.\rangle)$ the expression $\|x\| = \langle x,x\rangle^{1/2}$ defines a norm on $X$.*

PROOF. Homogeneity follows from the linearity of the innerproduct. The triangle inequality requires some work:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|,$$

so the right side is $(\|x\| + \|y\|)^2$, where we applied Cauchy-Schwarz to bound the innerproduct in terms of the norms of its elements. Thus we have $\|x+y\| \leq \|x\| + \|y\|$. □

EXAMPLE 2.3.5. (1) The sequence space $\ell^2$ is an innerproduct space for real-valued sequences $(x_i), (y_i)$

$$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i y_i.$$

The sequence space $\ell^2$ was the first example of an innerproduct space, studied by D. Hilbert in 1901 in his work on Fredholm operators.
Hölder's inequality for $p = 2$ gives $|\langle x,y\rangle| \leq \|x\|_2 \|y\|_2$, which is the Cauchy-Schwarz inequality in this case.

(2) The 2-norm $\|.\|_2$ for the space of continuous functions on the interval $C[a,b]$ is inducded from the innerproduct

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx.$$

The Cauchy-Schwarz inequality for $(C(\mathbb{R}), \langle .,.\rangle$ is due to Karl H. A. Schwarz in 1888.

The innerproduct $\langle .,.\rangle$ and its associated norm $\|.\| = \langle .,.\rangle^{1/2}$ are related by the *polarization identity.*

LEMMA 2.17 (Polarization identity). *Let $(X, \langle .,.\rangle)$ be an innerproduct space with norm $\|.\| = \langle .,.\rangle^{1/2}$. For a real innerproduct space we have $\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ for all $x,y \in X$.*

PROOF. The arguments are based on the properties of innerproducts. $\|x+(-1)^k y\|^2 = \|x\|^2 + \|y\|^2 + (-1)^k \langle x, y\rangle$ for $k = 0, 1$. Adding these two identities yields the desired polarization identity. □

Jordan and von Neumann gave an elementary characterizations of norms that arise from innerproducts.

THEOREM 2.18 (Jordan-von Neumann). *Suppose $(X, \|.\|)$ is a complex normed space. If the norm satisfies the parallelogram identity*

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad forall\ x, y \in X,$$

*then $X$ is an innerproduct space for the innerproduct*

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2.$$

PROOF. One direction is just a computation like the one done for the polarization identity. The reverse direction is based on defining an innerproduct in terms of the norms by turning the parallelogram identity into a definition and show that this is indeed an innerproduct. In the course of the argument one takes advantange of the paralellogram identity. □

Innerproduct spaces are the infinite-dimensional counterparts of $(\mathbb{R}^n, \|.\|_2)$ and share many properties with these finite-dimensional spaces, in contrast to general normed spaces such as $C(I)$ with the supnorm or $\ell^p$ for $p \neq 2$.

EXAMPLE 2.3.6. The supremum norm of $C[0, 1]$ does not come from an innerproduct. Use the polarization identity to show this fact.

We consider the case of complex innerproduct spaces that are of relevance in quantum mechanics and signal analysis as well as mathematics.

For vectors in $\mathbb{C}^2$ we have the ‘dot product‘ aka ‘scalar product‘ that assigns to a pair of vectors $z = (z_1, z_2)$ and $z' = (z'_1, z'_2)$ the complex number

$$\langle z, z'\rangle = z_1\overline{z_1}' + z_2\overline{z_2}'.$$

The reason for adding the complex conjugates to the definition of the real case is to get the length of $z = (z_1, z_2) \in \mathbb{C}^2$:

$$\|z\|^2 = \langle z, z\rangle = z_1\overline{z_1} + z_2\overline{z_2} = |z_1|^2 + |z_2|^2.$$

DEFINITION 2.3.7. Let $X$ be a complex vector space. An *innerproduct* on $X$ is a map $\langle ., .\rangle : X \times X \to \mathbb{C}$ satisfying:

(1) (Linearity) For vectors $x_1, x_2, y \in X$ and scalars $\alpha_1, \alpha_2 \in \mathbb{F}$ we have $\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = \alpha_1 \langle x_1, y\rangle + \alpha_2 \langle x_2, y\rangle$.
(2) (Conjugate Symmetry) For vectors $x, y \in X$ we have $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
(3) (Positive definiteness) For any $x \in X$ we have $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

We call $(X, \langle ., .\rangle)$ an *innerproduct space* and define by $\|x\| := \langle x, x\rangle^{1/2}$.

Note that the conjugate symmetry and linearity in the first entry gives that $\langle ., .\rangle$ is conjugate linear in the second entry: For vectors $x, y_1, y_2 \in X$ and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ we have $\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle\, y = \overline{\alpha_1}\langle x, y_1\rangle + \overline{\alpha_2}\langle x, y_2\rangle$.

Proposition 2.3.8 (Cauchy-Schwarz). *Suppose $X$ is a complex innerproduct space. Then for all $x, y \in X$ we have*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

*We have $|\langle x, y\rangle| = \|x\|\|y\|$ if and only if $x = \alpha y$ for some $\alpha \in \mathbb{C}$.*

Proof. Suppose $x$ and $y$ are non-zero vectors of $X$.

$$\begin{aligned} 0 &\leq \langle x - y, x - y\rangle = \langle x, x\rangle + \langle y, y\rangle - \langle y, x\rangle - \langle x, y\rangle \\ &= \langle x, x\rangle + \langle y, y\rangle - 2\mathrm{Re}\,\langle x, y\rangle, \end{aligned}$$

and we obtain an additive inequality:

$$\mathrm{Re}\,\langle x, y\rangle \leq \frac{1}{2}\langle x, x\rangle + \frac{1}{2}\langle y, y\rangle.$$

The normalization method turns this one into a multiplicative one: We set $\tilde{x} = x/\langle x, x\rangle^{1/2}$ and $\tilde{y} = y/\langle y, y\rangle^{1/2}$ and plug $\tilde{x}$ and $\tilde{y}$ into the preceding inequality:

$$\mathrm{Re}\,\langle x, y\rangle \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

We want to have a bound on $|\langle x, y\rangle|$ based on the one on the real part of $\langle x, y\rangle$ via pre-multiplication. By the later one means that one premultiplies by a well-chosen complex number in order to guarantee that some quantity will be real. In our case we use the polar decomposition of $\langle x, y\rangle$: $\langle x, y\rangle = |\langle x, y\rangle|e^{i\varphi}$ for some $\varphi \in [0, 2\pi)$. We set $\tilde{x} := e^{-i\varphi}x$

$$|\langle x, y\rangle| = \mathrm{Re}\tilde{x}y \leq \langle \tilde{x}, \tilde{x}\rangle^{1/2}\langle y, y\rangle^{1/2} = \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2},$$

which yields the complex Cauchy-Schwarz inequality. The case of equality is a consequence of the argument. □

EXAMPLE 2.3.9. (1) The space $\ell^2$ of square-integrable complex-valued sequences $(z_i), (z_i')$ is an innerproduct space:

$$\langle z, z' \rangle = \sum_{i=1}^{\infty} z_i \overline{z_i'}.$$

Hölder's inequality for $p = 2$ gives $|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$, which is the Cauchy-Schwarz inequality in this case.

(2) The 2-norm $\|.\|_2$ for the space of continuous complex-valued functions on the interval $C[a, b]$ is induced from the innerproduct

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

This innerproduct is of utmost importance in Schrödinger's approach to quantum mechanics and in signal analysis. In physics one often denotes $\langle f, g \rangle$ by $\langle f|g \rangle$ and they tend to have it conjugate linear in the first entry and linear in the second.

By the same reasoning as for real innerproduct spaces $X$ we deduce that $\|z\| := \langle z, z \rangle^{1/2}$ is a norm on $X$ Innerproducts provide a generalization of the notion of *orthogonality* of elements.

DEFINITION 2.3.10. Two elements $x, y$ in an innerproduct space $(X, \langle ., , \rangle)$ are *orthogonal* to each other if $\langle x, y \rangle = 0$

The theorem of Pythagoras is true for any innerproduct space $(X, \langle ., . \rangle)$.

PROPOSITION 2.3.11 (Pythagoras's Theorem). *Let $(X, \langle ., . \rangle)$ be an innerproduct space. For two orthogonal elements $x, y \in X$ we have*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

PROOF. The argument is based on the fact that $\langle x, x \rangle$ is a norm. By assumption we have $\langle x, y \rangle = 0$

$$\|x + y\|^2 = \|x\|^2 + 2\mathrm{Re}\,\langle x, y \rangle + \|y\|^2 = \|z\|^2 + \|y\|^2.$$

□

As an example we consider some orthogonal vectors in $(C([0, 1]), \langle ., . \rangle$. For $m \neq n$ we define the exponentials $e_m(x) = e^{2\pi imx}$ and $e_n(x) = e^{2\pi inx}$. Then

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i(m-n)x}dx = (2\pi i(m-n))^{-2}(e^{2\pi i(m-n)} - 1) = 0.$$

Note that $\langle e_n, e_n \rangle = 1$ for any $n \in \mathbb{Z}$. With the help of Kronecker's delta function we may express this as $\langle e_m, e_n \rangle = \delta_{m,n}$.

The theorem of Pythagoras is now at our disposal in any innerproduct spaces such as $\ell^2$.

Definition 2.3.12. A set of vectors $\{e_i\}_{i\in I}$ in an innerproduct space $(X,\langle.,,\rangle)$ is called an *orthogonal family* if $\langle e_i,e_j\rangle = 0$ for all $i\neq j$. In case that the orthogonal family $\{e_i\}_{i\in I}$ in $X$ satisfies in addition $\|e_i\| = 1$ for any $i\in I$, then we refer to it as *orthonormal family.*

The exponentials $\{e^{2\pi nx}\}_{n\in\mathbb{Z}}$ is an orthonormal family in $C[0,1]$ with respect to $\langle.,.\rangle$ and is a system of utmost importance, e.g. it lies at the heart of Fourier analysis or more generally harmonic analysis.

CHAPTER 3

# Banach and Hilbert spaces

We extend the topological notions introduced for the real line to general normed spaces and we focus on completeness in this section. Complete normed spaces are nowadays called Banach spaces, after the numerous seminal contributions of the Polish mathematician Stefan Banach to these objects. The class of complete innerproduct spaces are named after David Hilbert, who introduced the sequence space $\ell^2$. His students made numerous contributions to the theory of innerproduct spaces, e.g. Erhard Schmidt, Hermann Weyl, Otto Toeplitz,... .

## 3.1. Sequences in normed spaces

Norms on a vector space are the tool that provides us with a way to merge linear algebra and analysis, which is known as functional analysis. We will discuss some of the basic aspects of functional analysis in this course. We start with the notion of convergent sequences and will work our way up to completeness.

Definition 3.1.1. Let $(X, \|.\|)$ be a normed space. A sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ is said to **converge to** $x \in X$ if for a given $\varepsilon > 0$ there exists a $N$ such that $\|x - x_n\| < \varepsilon$ for $n \geq N$. The vector $x$ is called the **limit** of the sequence $(x_n)_{n\in\mathbb{N}}$.

Suppose $A$ is a subset of $X$. Given a convergent sequence $(a_n)_{n\in\mathbb{N}}$ in $A$, meaning all the $a_n$'s are elements of $A$. Then the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ is also known as a *limit point of* $A$. We denote the union of $A$ and all its limit points by $\overline{A}$.

This notion of convergence for sequences in normed spaces is a natural generalization of the one for real and complex numbers. Note that the elements of the sequences are vectors in a normed space. For example, a sequence in $\ell^2$ is a sequence where the elements themselves are also sequences. A more geometric view towards this notion of convergence is that for any $\varepsilon > 0$ there exists an $N$ such that $(x_N, x_{N+1}, ...)$ lies in the ball, $B_\varepsilon(x)$, of radius $\varepsilon$ around the limit $x$. Sometimes $(x_N, x_{N+1}, ...)$ is called the **tail** of the sequence $(x_n)_{n\in\mathbb{N}}$. Hence convergence of $x_n \to x$ means that for arbitrary small balls around the limit

$x$ the tail of $(x_n)_{n\in\mathbb{N}}$ lies in $B_\varepsilon(x)$.

Note that $x \in \overline{A}$ if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $A$ such that $a_n \to x$.

LEMMA 3.1. *Suppose the sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \|.\|)$ converges to a $x \in X$. Then*

$$\Big|\, \|x_n\| - \|x\| \,\Big| \to 0.$$

PROOF. By assumption we have that for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq N$. By the reverse triangle inequality we have that

$$\Big|\, \|x_n\| - \|x\| \,\Big| \leq \|x_n - x\|$$

but the right hand side goes to zero by the convergence of $(x_n)$ and thus we have that $\|x_n - x\| \to 0$. □

The notion of convergence depends on the norm the vector space is equipped with!

EXAMPLE 3.1.2. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ in $C[0,1]$ defined by $f_n(t) = e^{-nt}$. Then we have that $f_n$ converges to 0 in $(C[0,1], \|.\|_1)$:

$$\|f_n - 0\|_1 = \int_0^1 e^{-nt}dt = \frac{1}{n}(1 - e^{-n}) \to 0$$

as $n \to \infty$. Let us now discuss the convergence of $(f_n)_{n\in\mathbb{N}}$ in $(C[0,1], \|.\|_\infty)$. Since $\|f_m\|_\infty = \sup_{t\in[0,1]} |e^{-nt}| = 1$, so $(f_n)_{n\in\mathbb{N}}$ does not converge to the zero function with respect to $\|.\|_\infty$.

This example has a further feature.

EXAMPLE 3.1.3. Let $A$ be the set of positive functions in $C[0,1]$, i.e. $A = \{f \in C[0,1] : f(t) > 0,\ t \in [0,1]\}$. Then the convergence of $(f_n)_{n\in\mathbb{N}}$ in $(C[0,1], \|.\|_1)$ of $(e^{-nt})_{n\in\mathbb{N}}$ to zero, gives us a sequence in $A$ with a limit not contained in $A$; the zero function is the very example of a function attaining zero in $[0,1]$.

As for real sequences we have that limits of convergent sequences are unique.

LEMMA 3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in the normed space $(X, \|.\|)$. Then its limit is unique.*

PROOF. Suppose there exist two limits $x, y$ of $(x_n)_{n\in\mathbb{N}}$. Then for any $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ $\|x_n - x\| \leq$

$\varepsilon/2$ and for all $n \geq N-2$ we have $\|x_n - y\| \leq \varepsilon/2$. Hence for all $n \geq \max N_1, N_2$ we have

$$\|x-y\| = \|x - x_n + x_n - y\| \leq \|x - x_n\| + \|x_n - y\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

A convergent sequence of real numbers is bounded, i.e. there exists a constant $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Convergent sequences in normed spaces are also bounded if one defines the boundedness of a subset of this space in an analogous manner.

Definition 3.1.4. A subset $A$ of $(X, \|.\|)$ is called **bounded** if $A$ is contained in some ball $B_{r_0}(x_0)$ for some radius $r_0$ and point $x_0 \in X$. In this case we define the **diameter** of $A$, $\mathrm{diam}(A)$, to be the real number $\sup\{\|x-y\| : x, y \in X\}$.

Let us state some reformulations of the notion of boundedness of a set.

Lemma 3.3. *For a subset $A$ of a normed space $X$ the following statements are equivalent:*

(1) *$A$ is bounded.*
(2) *There exists a constant $M > 0$ such that $\|x-y\| \leq M$ for all $x, y \in A$.*
(3) $\mathrm{diam}(A) < \infty$
(4) *For every $x \in X$ there exists a radius $r > 0$ such that $A \subseteq B_r(x)$.*
(5) *There exists a $m > 0$ such that $\|x\| \leq m$ for all $x \in A$.*

Proof. We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$, and finally $(v) \Rightarrow (i)$.

If (i) holds, then for some $x_0 \in X$ and $r_0 > 0$ we have $A \subset B_{r_0}(x_0)$:

$$\|x-y\| \leq \|x - x_0\| + \|x_0 - y\| \leq 2r_0 \text{ for all } x, y \in A,$$

i.e. $\|x-y\| \leq M = 2r_0$ for all $x, y \in A$.

If (ii) holds, then by the definition of supremum, as least upper bound, of the set $\{\|x-y\| : x, y \in A\}$ is less than or equal to the finite constant $M$, i.e. the diameter of $A$ is finite.

If (iii) holds, then for all $x, y \in A$ we have $\|x-y\| \leq \mathrm{diam}(A) < \infty$. Choose an element $a_1 \in A$. Then given any $x \in X$ and $a \in A$ we have $\|x-a\| \leq \|x - x_1\| + \|x_1 - a\| \leq d(x, a_1) + \mathrm{diam}(A) =: r$ and $A \subseteq B_r(x)$. Hence we have shown that $(iii) \Rightarrow (iv)$.

The assertion $(iv) \Rightarrow (i)$ by definition of boundedness.

If $(v)$ holds, then $A \subset B_m(0)$. Thus we have $A$ is contained in a ball of radius $m$ around the origin which is possible since in vector spaces we can translate its elements by a given vector such that the set gets centered at the origin. $\square$

Further results about boundedness are posed as problems on the next problem set: (i) Any ball $B_r(x) \subset (X, \|.\|)$ is bounded and $\mathrm{diam}(B_r(x)) \leq 2r$. (ii) If $A$ is a bounded subset, then for any $a \in A$ we have $A \subseteq B_{\mathrm{diam}(A)}(a)$.

LEMMA 3.4. *A convergent sequence in a normed space $X$ is bounded.*

PROOF. See problem set. $\square$

The definition of convergence of a sequence has one flaw: Namely one needs to have a candidate for the limit beforehand to actually set up the proof that the sequence converges to this particular object. Cauchy has noted that it is much more suitable to have a condition that only involves the sequence elements.

DEFINITION 3.1.5. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(X, \|.\|)$. Then we call $(x_n)_{n\in\mathbb{N}}$ a **Cauchy sequence** if for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$\|x_n - x_m\| < \varepsilon.$$

LEMMA 3.5. *Any Cauchy sequence in $(X, \|.\|)$ is bounded.*

PROOF. See problem set. $\square$

LEMMA 3.6. *Every convergent sequence in $(X, \|.\|)$ is a Cauchy sequence.*

PROOF. Let $x_n \to x$ in $(X, \|.\|)$. Then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon/2$ for all $n \geq N$. Hence for $m, n \geq N$ we have

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| \leq \varepsilon.$$

$\square$

EXAMPLE 3.1.6. We define a sequence in $(C[a,b], \|.\|_1)$ by a sequence of piece-wise continuous functions $f_n$:

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

$(f_n)$ is a Cauchy sequence in $(C[.a,b], \|.\|_1)$.
For $m > n$ the slope of $f_m$ is greater than of $f_n$ and thus the area of the function $f_m - f_n$ can be bounded by the triangle with sides 1 and $1/n$, i.e. $\|f_m - f_n\|_1 \leq 1/2n$.

There are Cauchy sequences in $(C[a,b], \|.\|_1)$ that have no continuous limit function.

PROPOSITION 3.1.7. $(C[a,b], \|.\|_1)$ *is not complete.*

PROOF. The sequence $(f_n)$ defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

is Cauchy sequence in $(C[a,b], \|.\|_1)$ with discontinuous limit function:

$$\begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ 1 & \text{for } \frac{a+b}{2} \leq t \leq b. \end{cases}$$

Suppose $f_n \to f$ in $\|.\|_1$ with $f \in C[a,b]$. Let us analyze the implications of $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

$$\int_a^b |f_n(t) - f(t)|dt = \Big[\int_a^{\frac{a+b}{2}} + \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} + \int_{\frac{a+b}{2}+\frac{1}{n}}^{b}\Big] |f_n(t) - f(t)|dt$$

breaks up into three integrals:

(1) $\int_a^{\frac{a+b}{2}} |f_n(t) - f(t)|dt \to 0$ only if $f = 0$ on $[a, \frac{a+b}{2}]$;

(2) $\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} |f_n(t) - f(t)|dt \to 0$. Since $f_n$ is continuous for all $n \in \mathbb{N}$ and $f$ is continuous on $[a,b]$ we have

$$\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} |f_n(t) - f(t)|dt \leq (\max_{t\in[0,1]} |f(t)| + 1)\frac{1}{n} \to 0$$

as $n \to \infty$. Hence this imposes no condition on the limit function $f$.

(3) By the continuity of $f$ we have that

$$\int_{\frac{a+b}{2}+\frac{1}{n}}^{b} |f_n(t) - f(t)|dt = \int_{\frac{a+b}{2}+\frac{1}{n}}^{b} |1 - f(t)|dt \to \int_{\frac{a+b}{2}}^{b} |1 - f(t)|dt,$$

as $n \to \infty$. Hence this limit is zero, we must have $1 - f(t) = 0$, i.e. $f(t) = 1$ for all $t \in [\frac{a+b}{2}, b]$.

In summary, the limit function $f$ on $[a,b]$ has a jump discontinuity at $\frac{a+b}{2}$. $\square$

## 3.2. Completeness

The difference between the the normed space $(\mathbb{Q}, |.|)$ and the real numbers $(\mathbb{R}, |.|)$ viewed as normed space is that not all Cauchy sequences in $\mathbb{Q}$ converge to a rational number but that is the case for $\mathbb{R}$. Cauchy established that any Cauchy sequence in $\mathbb{R}$ converges and its limit is again a real number. In order to show this we assume a property of the set of real numbers without proof, a so-called axiom. Namely, $\mathbb{R}$ is supposed to have the **least upper bound property: Any non-empty subset $S$ that is bounded from above has a supremum $\sup S$ and $\sup S$ is a real number.**

For example the set $\{a \in \mathbb{Q} : \ a < \sqrt{3}\}$ is bounded above by $\sqrt{3}$, but $\sqrt{3}$ is not a rational number. We include the proof of this important fact.

Proposition 3.2.1. *The equation*

$$x^2 - 3 = 0$$

*has no solutions in* $\mathbb{Q}$.

Proof. We assume by contradiction that there is a rational number $r$ such that $r^2 - 3 = 0$.

We represent $r$ as a *reduced* fraction. That is, we write $r = \frac{p}{q}$ where $p, q$ are integers, $q \neq 0$ and $\gcd(p,q) = 1$. We then have:

$$r^2 - 3 = 0 \implies r^2 = 3 \implies \frac{p^2}{q^2} = 3 \implies p^2 = 3\,q^2.$$

The last identity says that $p^2$ is a multiple of 3. Then $p$ itself must be a multiple of 3 as well (why?), which means that $p = 3m$ for some integer $m$.

Substituting this into the identity $p^2 = 3q^2$ we get $9m^2 = 3q^2$, which implies $3m^2 = q^2$, and so $q^2$ must be a multiple of 3. But then $q$ must also be a multiple of 3.

Let us step back and look at what we have: we started of with a completely reduced fraction $r = \frac{p}{q}$, assumed that $r^2 - 3 = 0$, which through a series of derivations led to the conclusion that both $p$ and $q$ must be multiples of 3. This contradicts the fraction $\frac{p}{q}$ being reduced.

Therefore, the equation $x^2 - 3 = 0$ cannot have any rational number as solution. $\square$

Theorem 3.7. *A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ converges if and only if for any $\varepsilon > 0$ there exists an index $N$ such that for all $m, n \in \mathbb{N}$ we have $|a_m - a_n| < \varepsilon$.*

Proof. The statement about convergent sequences satisfying the Cauchy property is one of the problems of problem set 5. The other implication is much more intricate. Suppose we have a Cauchy sequence $(a_n)_{n=1}^{\infty}$. Then we claim it converges to a real number. The argument is elementary but a little bit involved. Let $A$ be the set of elements of our sequence $(a_n)$, $A = \{a_1, a_2, ...\}$. Then $A$ is a bounded subset of $\mathbb{R}$: there exists an $M > 0$ such that $a_n \in [-M, M]$ for $n = 1, 2, ...$ . Take $\varepsilon = 1$ in the Cauchy condition: Then there exists an integer $N_1$ such that for all $m, n \geq N_1$ such that $|a_n - a_{N_1}| < 1$ and thus the set $\{a_1, a_2, ..., a_{N_1}, a_{N_1+1}\}$ is bounded by a constant $M$.

Now we consider the set

$$S := \{s \in [-M, M] : \text{there exist infinitely many } n \in \mathbb{N} \text{ for which } a_n \geq s\},$$

in other words we collect all the numbers $s$ in $[-M, M]$ such that $a_n \geq s$ infinitely often. Definitely $-M \in S$ and $S$ is bounded above by $M$. Thus by the least upper bound property of $\mathbb{R}$ there exists a real number $a$ such that $a = \sup S$.

**Claim:** $a_n \to a$ as $n \to \infty$.
For any $\varepsilon > 0$ the Cauchy condition provides an $N_2$ s.t. for all $m, n \geq N_2$:

$$|a_m - a_n| < \varepsilon/2.$$

All elements of $S$ are less than or equal to $a$, so the larger number $a + \varepsilon/2$ does not belong to $S$, and hence only finitely many often does $a_n$ exceed $a + \varepsilon/2$. That is for some $N_3 \geq N_2$ we have for all $n \geq N_3$ that

$$a_n \geq a + \varepsilon/2.$$

Since $a$ is a least upper bound for $S$, the smaller number $a - \varepsilon/2$ cannot also be an upper bound for $S$. Hence, there is some $s \in S$ such that $s \geq a - \varepsilon/2$. Consequently, we have infinitely many sequence elements such that

$$a - \varepsilon/2 < s \leq a_n.$$

In particular, there exists an $N \geq N_3$ such that

$$a_N > a - \varepsilon/2$$

. Since $N \geq N_3$ we have $a_N \leq a + \varepsilon/2$ and so $a_N \in (a - \varepsilon/2, a + \varepsilon/2)$. Now recall that $N \geq N_2$ which yields that

$$|a_n - a| \leq |a_n - a_N| + |a_N - a| < \varepsilon$$

for all $n \geq N$, i.e. $a_n \to a$ as $n \to \infty$.

$\square$

The property of $\mathbb{R}$ that any Cauchy sequence converges in $\mathbb{R}$ is a favorable property that we would like to have for general normed spaces.

Definition 3.2.2. A normed space $(X, \|.\|)$ is called *complete* if every Cauchy sequence $(x_k)$ in $X$ has a limit $x$ belonging to $X$. Moreover, a complete normed space is referred to as *Banach space* and a complete innerproduct space is known as *Hilbert space*.

Let us start with an elementary observation that is a straightforward consequence of the definitions.

Theorem 3.8. *$(\mathbb{R}^n, \|.\|_\infty)$ is a Banach space.*

The completeness of the normed space $(\mathbb{R}, |.|)$ has numerous ramifictions.

Proof. The $\|.\|_\infty$-convergence of $(x_n)_{n\in\mathbb{N}}$ implies the coordinate wise convergence. Since any Cauchy sequence in $(\mathbb{R}^n, \|.\|_\infty)$ gives Cauchy sequences in each coordinate. Since $\mathbb{R}$ is complete we deduce that all these coordinate Cauchy sequences converge in $\mathbb{R}$. Thus we have that $(\mathbb{R}^n, \|.\|_\infty)$ is complete. $\square$

Theorem 3.9. *The space of absolutely summable sequences is a Banach space with respect to $\|.\|_1$-norm; i.e. $(\ell^1, \|.\|_1)$ is a Banach space.*

Proof. The argument is split into three steps.
*Step 1: Find a candidate for the limit.* Let $(x_n)_n$ be a Cauchy sequence in $\ell^1$. We denote the n-th element of the sequence by $x_n = (x_1^{(n)}, x_2^{(n)}, ...)$.
Note that $|x_1^{(m)} - x_1^{(n)}| \leq \|x_m - x_n\|_1$, so the first coordinates $(x_1^{(n)})_n$ are a Cauchy sequence of real numbers and hence converge to some real number $z_1$. Similarly, the other coordinates converge: $z_j = \lim_{n\to\infty} x_j^{(n)}$. Hence our candidate for the limit of $(x_n)$ is the sequence $z = (z_1, z_2, ...)$.
*Step 2: Show that $z$ is in $\ell^1$.* We have that

$$\sum_{j=1}^{N} |z_j| = \sum_{j=1}^{N} \lim_n |x_j^{(n)}| = \lim_n \sum_{j=1}^{N} |x_j^{(n)}|,$$

where the interchange of the limit with the sum of a finite number of real numbers is no problem. Since Cauchy sequences are bounded, there is a constant $C > 0$ such that $\|x_n\|_1 < C$ for all $n$. Thus for any $N$

$$\sum_{j=1}^{N} |x_j^{(n)}| \leq \sum_{j=1}^{\infty} |x_j^{(n)}| = \|x_n\|_1 < C.$$

Letting $n \to \infty$ we find that

$$\sum_{j=1}^{N} |z_j| \leq \|x_n\|_1 < C$$

for arbitrary $N$. Hence we have $z \in \ell^1$.
*Step 3: Show the convergence.* We want to prove that $\|x_n - z\|_1 \to 0$ for $n \to \infty$.
Given $\varepsilon > 0$, pick $N_1$ so that if $m, n > N_1$ then $\|x_m - x_n\|_1 < \varepsilon$. Hence for any fixed $N$ and $m, n > N_1$, we find

$$\sum_{j=1}^{N} |x_j^{(m)} - x_j^{(n)}| \leq \sum_{j=1}^{\infty} |x_j^{(m)} - x_j^{(n)}| = \|x_n - x_m\| < \varepsilon.$$

Fix $n > N_1$ and $N$, let $m \to \infty$ to obtain

$$\sum_{j=1}^{N} |x_j^{(n)} - z_j| = \lim_{n\to\infty} |x_j^{(n)} - x_j^{(m)}| \leq \varepsilon.$$

Since this is true for all $N$ we have demonstrated that

$$\|x_n - z\|_1 < \varepsilon.$$

That is our desired conclusion. □

THEOREM 3.10. *The space of bounded sequences is a Banach space with respect to $\|.\|_\infty$-norm; i.e. $(\ell^\infty, \|.\|_\infty)$ is a Banach space.*

PROOF. The argument is once more split into three steps.
*Step 1: Find a candidate for the limit.* Let $(x_n)_n$ be a Cauchy sequence in $\ell^\infty$. We denote the n-th element of the sequence by $x_n = (x_1^{(n)}, x_2^{(n)}, ...)$.
Note that $|x_k^{(m)} - x_k^{(n)}| \leq \|x_m - x_n\|_\infty$ for all $k$ and all $m, n > N$, so the k-th coordinates $(x_k^{(n)})_n$ are a Cauchy sequence of real numbers and hence converge to some real number $z_k$. Similarly, the other coordinates converge: $z_k = \lim_{m\to\infty} x_k^{(n)}$.

Hence our candidate for the limit of $(x_n)$ is the sequence $z = (z_1, z_2, ...)$.
*Step 2: Show that $z$ is in $\ell^\infty$.* We have that

$$\sup\{|z_j| : j = 1, ..., N\} = \sup\{\lim_n |x_j^{(n)}| j = 1, ..., N\} = \lim_n \{\sup |x_j^{(n)}| j = 1, ..., N\},$$

where the interchange of the limit with the sum of a finite number of real numbers is no problem. Since Cauchy sequences are bounded, there is a constant $C > 0$ such that $\|x_n\|_\infty < C$ for all $n$. Thus for any $N$

$$\lim_n \{\sup |x_j^{(n)}| j = 1, ..., N\}| \leq \|x_n\|_\infty < C.$$

Thus we find that $\|x_n\|_\infty < C$, i.e. we have $z \in \ell^\infty$.
*Step 3: Show the convergence.* We want to prove that $\|x_n - z\|_\infty \to 0$ for $n \to \infty$.
Given $\varepsilon > 0$, pick $N_1$ so that if $m, n > N_1$ then

$$|x_m^{(k)} - x_n^{(k)}| \leq \|z_k - x_n^{(k)}\|_\infty < \varepsilon$$

for all $k$. Taking limits as $m \to \infty$ we have

$$|z_k - x_n^{(k)}| \leq \varepsilon$$

Taking supremum in $k$, we obtain

$$\sup_k |z_k - x_n^{(k)}| \leq \varepsilon$$

for all $n > N_1$, i.e. $\|x_n - z\|_\infty \leq \varepsilon$ for all $n > N$. Consequently we have that $x_n$ converges to $z$ in $(\ell^\infty, \|.\|_\infty)$.

□

Reasoning similar to the one for $\ell^1$ gives us that all $\ell^p$-spaces are Banach spaces for $\|.\|_p$ when $1 \leq p < \infty$.

THEOREM 3.11. *Let $[a, b]$ be a bounded interval of real numbers. Then the normed space $C[a, b]$ with respect to the sup-norm $\|.\|_\infty$ is a Banach space.*

The situation is different for the function spaces $(C[a, b], \|.\|_p)$, as we have seen before for $p = 1$ this is not a complete space and this is also true for $1 \leq p < \infty$. In contrast $(C[a, b], \|.\|_\infty)$ is a complete space. Before we are able to proof this statement we have to discuss different notions of convergence for sequences of functions and properties of continuous functions.

LEMMA 3.12. *For $f, g \in C[a, b]$ we have that* $\sup\{|f(x) - g(x)| x \in [a, b]\}$ *is finite, and there is a $y \in [a, b]$ such that $d_\infty(f, g) = |f(y) - g(y)| = \max\{|f(x) - g(x)| x \in [a, b]\}$.*

Proof. We show that $d(x) = |f(x) - g(x)|$ is continuous on $[a, b]$ and thus by the Extreme Value Theorem the assertion follows. The continuity of $d$ is deduced from

$$|d(x)-d(y)| \leq ||f(x)-g(x)|-|f(y)-g(y)|| \leq |f(x)-f(y)|+|g(y)-g(x)|.$$

Since $f$ and $g$ are continuous at $x$ there is for any given $\varepsilon > 0$ a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$ for $|x - y| < \delta$. Hence

$$|d(x) - d(y)| \leq |f(x) - f(y)| + |g(y) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $y \in [a, b]$ with $|x - y| < \delta$. Consequently $d$ is continuous. □

Remark 3.2.3. Observe that the $\|f - g\|_\infty$-norm measures the distance between the functions $f$ and $g$ by looking at the point in $a[a, b]$ they are the furthest apart.

Definition 3.2.4. Let $(f_n)$ be a sequence of functions on a set $X$.

- We say that $(f_n)$ *converges pointwise* to a limit function $f$ if for a given $\varepsilon > 0$ and $x \in X$ there exists an $N$ so that
$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N.$$
- We say that $(f_n)$ *converges uniformly* to a limit function $f$ if for a given $\varepsilon > 0$ there exists an $N$ so that
$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N$$
holds for all $x \in X$.

There is a substantial difference between these two definitions. In pointwise convergence, one might have to choose a different $N$ for each point $x \in X$. In the case of uniform convergence there is an $N$ that holds for all $x \in X$. Note that uniform convergence implies pointwise convergence. If one draws the graphs of a uniformly convergent sequence, then one realizes that the definition amounts for a given $\varepsilon > 0$ to have a $N$ so that the graphs of all the $f_n$ for $n \geq N$, lie in an $\varepsilon$-band about the graph of $f$. In other words, the $f_n$'s get uniformly close to $f$. Hence uniform convergence means that the maximal distance between $f$ and $f_n$ goes to zero. We prove this assertion in the next proposition.

Proposition 3.2.5. *Let $(f_n)$ be a sequence of continuous functions on $[a, b]$. Then the following are equivalent:*

(1) *$(f_n)$ converges uniformly to $f$.*
(2) $\sup\{|f_n(x) - f(x)| : x \in [a, b]\} \to 0$ *as* $n \to \infty$.

PROOF. Assertion $(i) \Rightarrow (ii)$: Assume that $(f_n)$ converges uniformly to $f$. Then for any $\varepsilon > 0$ there exists a $N$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ and all $n > N$. Hence $\sup\{|f_n(x) - f(x)| : x \in [a,b]\} \leq \varepsilon$ for all $n > N$. Since this holds for all $\varepsilon > 0$, we have demonstrated that $\sup\{|f_n(x) - f(x)| : x \in [a,b]\} \to 0$ for $n \to \infty$.
Assertion $(ii) \Rightarrow (i)$: Assume that $\sup\{|f_n(x) - f(x)| : x \in [a,b]\} \to 0$ for $n \to \infty$. Given an $\varepsilon > 0$, there is a $N$ such that $\sup\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon$ for all $n > N$. Thus we have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ and all $n > N$, i.e. $(f_n)$ converges uniformly to $f$. $\square$

A reformulation of this result is that a sequence converges in $(C[a,b], \|.\|_\infty)$ to $f$ is equivalent to the uniform convergence of $(f_n)$ to $f$.

PROPOSITION 3.2.6. *A sequence $(f_n)$ converges to $f$ in in $(C[a,b], \|.\|_\infty)$ if and only if $(f_n)$ converges uniformly to $f$.*

Uniform convergence has an important property.

THEOREM 3.13. *Let $(f_n)$ be a uniformly convergent sequence in $C[a,b]$ with limit $f$. Then the limit function $f$ is continuous on $[a,b]$.*

PROOF. Let $y \in I$ and $\varepsilon > 0$ be given. By the uniform convergence of $f_n \to f$, there exists an $N$ such that $n \geq N$ implies that

$$|f_n(x) - f(x)| \leq \varepsilon/3 \quad \text{for all } x \in I.$$

The continuity of $f_N$ implies that there exists a $\delta > 0$ such that

$$|f_N(x) - f(y)| \leq \varepsilon/3 \quad \text{for } |x - y| \leq \delta.$$

We want to show that $f$ is continuous. For all $x$ such that $|x - y| < \delta$ we have that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

$\square$

THEOREM 3.14. $(C[a,b], \|.\|_\infty))$ *is a Banach space.*

PROOF. Convergence of a sequence in $(C[a,b], \|.\|_\infty)$ to $f \in C[a,b]$ is equivalent to uniform convergence of the sequence to $f$.

Assume that $(f_n)$ is a Cauchy sequence in $(C[a,b], \|.\|_\infty)$. Then we have to show that there exists a function $f \in C[a,b]$ that has $(f_n)$ as its limit.
Fix $x \in [a,b]$ and note that $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$. Since $(f_n)$ is a Cauchy sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(f_n(x))$ converges to a point $f(x)$ in $\mathbb{R}$. In other words,

$f_n \to f$ pointwise.
Next we show that $f \in C[a,b]$. Since $(f_n)$ is a Cauchy sequence, we have for any $\varepsilon > 0$ a $N$ such that $\|f_n - f_m\| < \varepsilon/2$ for all $m, n > N$. Hence we have $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $x \in [a,b]$ and for all $m, n > N$. Letting $m \to \infty$ yields for all $x \in [a,b]$ and all $n > N$:

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \varepsilon/2 < \varepsilon.$$

Consequently, $f_n \to f$ converges uniformly. Now by the preceding proposition $f$ is a continuous function on $[a,b]$. In other words, we have established that $(C[a,b], \|.\|_\infty)$ is a Banach space. □

## 3.3. Banach's Fixed Point Theorem

In 1922 Banach established a theorem on the convergence of iterations of contractions that has become a powerful tool in applied and pure mathematics aka Contraction Mapping Theorem. Before we state Banach's fixed point theorem we define continuous functions between normed spaces. A natural and far-reaching generalization of the notion of continuous functions defined on $\mathbb{R}$.We will have much more to say about continuous functions in the next chapter.

Definition 3.3.1. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces, let $A \subset X$ and let $f\colon A \to Y$ be a function.

(1) We say that $f$ is *continuous* at a point $a \in A$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in A$ with $\|x - a\| < \delta$ we have $\|f(x) - f(a)\| < \varepsilon$.
(2) We say that $f$ is *continuous* on $A$ if it is continuous at each point of $A$. Here is a useful criterion for continuity of a function.

A class of continuous functions on normed spaces is given by functions satisfying: There exists a finite constant $L$ such that

$$\|f(x) - f(x')\| \le L\,\|x - x'\| \quad \text{for all } x, x' \in A.$$

One calls such functions **Lipschitz continuous**, after the German mathematician R. Lipschitz, and often one refers to $L$ as **Lipschitz constant.** On Problem set 6 you will show that any Lipschitz continuous function is continuous.

We have come across Lipschitz continuous functions in our discussion of normed spaces. Namely, the reverse triangle inequality shows that a norm $\|,\| : X \to \mathbb{R}$ on a vector space $X$ is Lipschitz continuous with constant 1.

Proposition 3.3.2. *Let $f\colon A \to Y$ be a function, where $A \subset X$ and $X, Y$ are normed spaces. Let $a \in A$. Then the following two statements are equivalent.*

*(i) $f$ is continuous at $a$.*

*(ii) For every sequence $(x_n) \subset A$, if $x_n \to a$ then $f(x_n) \to f(a)$.*

Proof. i) $\Rightarrow$ (ii): We assume that $f$ is continuous at $a$.

Let $(x_n) \subset A$ be a sequence such that $x_n \to a$. We prove that $f(x_n) \to f(a)$.

Let $\varepsilon > 0$. Since $f$ is continuous at $a$, there is $\delta > 0$ such that if $\|x - a\| < \delta$ then $\|f(x) - f(a)\| < \varepsilon$.

Since $x_n \to a$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|x_n - a\| < \delta$. From the above, if $n \geq N$ we must then have $\|f(x_n) - f(a)\| < \varepsilon$.

As $\varepsilon$ was arbitrary, this proves that $f(x_n) \to f(a)$.

(i) $\Leftarrow$ (ii): We assume by contradiction that $f$ is *not* continuous at $a$. Let us write down carefully what that means.

Firstly, we recall the definition of continuity. $f$ is continuous at the point $a \in A$ means:
for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in A$ with $\|x - a\| < \delta$ we have $\|f(x) - f(a)\| < \varepsilon$.

Next, we formulate the *negation* of this statement.

The function $f$ is *not* continuous the point $a \in A$ means:
there is $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is an element of $A$, which we denote by $x_\delta$, such that $\|x_\delta - a\| < \delta$ but $\|f(x_\delta) - f(a)\| \geq \varepsilon_0$.

For every $n \geq 1$, we may choose $\delta = \frac{1}{n}$. Then for some element of $A$, which we denote by $x_n$, we have that $\|x_n - a\| < \frac{1}{n}$ but $\|f(x_n) - f(a)\| \geq \varepsilon_0$.

We have thus obtained a sequence $(x_n) \subset A$ such that $\|x_n - a\| < \frac{1}{n} \to 0$, so $x_n \to a$. However, since $\|f(x_n) - f(a)\| \geq \varepsilon_0$, the sequence $f(x_n) \not\to f(a)$, which is a contradiction.

Hence $f$ must be continuous at $a$. $\square$

Suppose we have a continuous function $f$ on a normed space $X$. Take a point $x_0$ in $X$ and build the sequence of iterates

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ \ldots,\ x_{n+1} = f(x_n).$$

The existence of the limit of this sequence $x = \lim_n x_n = \lim_n f^n(x_0)$ is the basic question that underlies Banach's fixed point theorem. The limit $x$ of the iterates $(x_n)$ is a fixed point of the continuous map $f$:

$$f(x) = f(\lim_n x_n) = \lim_n f(x_n) = \lim_n x_{n+1} = \lim x_n = x.$$

A mapping $f$ on a normed space $X$ is called a **contraction** if there exists a $0 < K < 1$ such that

$$\|f(x) - f(y)\| \leq K\|x - y\| \quad x, y \in X,$$

a contraction is a Lipschitz continuous function with Lipschitz constant $L < 1$. Recall that $\|x - y\| = d(x, y)$ is the distance between $x$ and $y$.

THEOREM 3.15 (Banach's Fixed Point). *Let $X$ be a Banach space $X$. Any contraction $f : X \to X$ has a unique fixed point $\tilde{x}$ and the fixed point is the limit of every sequence generated from an arbitrary nonzero point $x_0 \in X$ by iteration $(x_n)_n$, where $x_{n+1} = f(x_n)$ for $n \geq 1$.*

PROOF. Let $x_0 \in X$ be arbitrary. Define $x_{n+1} = f(x_n)$ for $n = 1, 2, \ldots$ . By the contractivity of $T$ we have

$$\|x_n - x_{n-1}\| = \|f(x_{n-1}) - f(x_{n-2})\| \leq K\|x_{n-1} - x_{n-2}\|$$

and iterations yields

$$\|x_n - x_{n-1}\| \leq K^{n-1}\|x_1 - x_0\|.$$

The existence of a fixed point is based on the completeness of $X$. Hence we proceed to show that $(x_n)_n$ is a Cauchy sequence. Let $m, n$ be greater than $N$ and we choose $m \geq n$. Then by the preceding inequality and the triangle inequality we have

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq (K^{m-1} + K^{m-2} + \cdots K^n)\|x_1 - x_0\| \\
&\leq (K^N + K^{N+1} + \cdots)\|x_1 - x_0\| \\
&= K^N(1 - K)^{-1}\|x_1 - x_0\|.
\end{aligned}$$

Since $0 \leq K < 1$, $\lim_N K^N = 0$ and thus $(x_n)$ is a Cauchy sequence. Consequently, $(x_n)$ converges to a point $\tilde{x}$ by the completeness of $X$. Furthermore $\tilde{x}$ is a fixed point by the contractivity of $T$.
Uniqueness: Suppose there is another fixed point $\tilde{y}$ of $f$. Then $\|\tilde{x} - \tilde{y}\| = \|f(\tilde{x}) - f(\tilde{y})\| \leq K\|\tilde{x} - \tilde{y}\|$ and $\|\tilde{x} - \tilde{y}\| > 0$. Thus we deduce that $K \geq 1$ which is a contradiction to $f$ being a contraction. □

Lipschitz maps with constant 1 are not eligible in this fixed point theorem. Since the map $f(x) = x + 1$ on $[0, 1]$ has no fixed point, but the map $f(x) = x$ on $[0, 1]$ has infinitely many fixed points.

COROLLARY 3.3.3. *Under the assumption in Banach's fixed point theorem we have the following estimates about the rate of convergence of the iterates $(x_n)$ towards the fixed point $\tilde{x}$:*

(1)
$$\|x_n - \tilde{x}\| \leq \frac{K^n}{1-K}\|x_0 - f(x_0)\|,$$
*tells us, in terms of the distance between $x_0$ and $f(x_0)$ how many times we need to iterate $f$ starting from $x_0$ to be certain that we are within a specified distance from the fixed point.*

(2)
$$\|x_n - \tilde{x}\| \leq K\|x_{n-1} - \tilde{x}\|,$$
*is called an a priori estimate, meaning that it gives us an upper bound on how long we need to compute to reach the fixed point.*

(3)
$$\|x_n - \tilde{x}\| \leq \frac{K}{1-K}\|x_{n-1} - x_n\|,$$
*tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This kind of estimate, called an a posteriori estimate, is very important because if two successive iterations are nearly equal, guarantees that we are very close to the fixed point.*

PROOF. From the proof we have that for $m > n$

$$\|x_m - x_n\| \leq \frac{K^n}{1-K}\|x_0 - x_1\| = \frac{K^n}{1-K}\|x_0 - f(x_0)\|. \tag{3.1}$$

The right side is *independent* of $m$ and so $m \to \infty$ gives

$$\|x_n - \tilde{x}\| \leq \frac{K^n}{1-K}\|x_0 - f(x_0)\|. \tag{3.2}$$

The second inequality comes along like that: Since $\tilde{x}$ is the unique fixed point of $f$:

$$\|x_n - \tilde{x}\| = \|f(x_n) - f(\tilde{x})\| \leq K\|x_{n-1} - \tilde{x}\|.$$

Applying the triangle inequality to $\|x_{n-1}-\tilde{x}\|$ gives the third inequality:

$$\|x_n - \tilde{x}\| \leq K(\|x_{n-1} - x_n\| + \|x_n - \tilde{x}\|),$$

which gives

$$\|x_n - \tilde{x}\| \leq \frac{K}{1-K}\|x_{n-1} - x_n\|. \tag{3.3}$$

□

Recall that we defined for a the closure $\overline{A}$ of $A$ as the union of $A$ and the set of limit points of $A$.

DEFINITION 3.3.4. A subset $A$ of $(X, \|.\|)$ is called **closed** if $\overline{A} = A$.

For example $\{y \in X : \|x-y\| \leq r\}$ is a closed subset of $X$. We will discuss properties of closed sets in the next chapter.

Here is a variant of Banach's fixed point theorem:

THEOREM 3.16. *Let $A$ be a closed subset of a Banach space $X$. If $f : A \to X$ is a contraction, then $f$ has a unique fixed point and the fixed point is the limit of every sequence generated from an arbitrary nonzero point $x_0 \in A$ by iteration $(x_n)_n$, where $x_{n+1} = f(x_n)$ for $n \geq 1$. If the contraction $f : A \to X$ satisfies in addition, $f(A) \subseteq A$, then the fixed point lies also in $A$.*

PROOF. See problem set. □

Two well-known applications are Newton's method for finding roots of general equations, solving systems of linear equations and the theorem of Picard-Lindelöf on the existence of solutions of ordinary differential equations. We discuss the first item and postpone the other items.

**Newton's method:**

How does one compute $\sqrt{3}$ up to a certain precision, i.e. we are interested in error estimates? Idea: Formulate it in the form $x^2-3=0$ and try to use a method that allows to compute zeros of general equations.

Newton came up with a method to solve $g(x) = 0$ for a differentiable function $g : I \to \mathbb{R}$.
Suppose $x_0$ is an approximate solution or starting point. Define recursively

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \quad \text{for } n \geq 0.$$

Then $(x_n)$ converges to a solution $\tilde{x}$, provided certain assumptions on $g$ hold.
If $x_n \to \tilde{x}$, then by continuity of $g$ we get $g(\tilde{x}) = 0$.

When does Netwon's method lead to a convergent sequence of iterates?
Idea: Apply Banach's Fixed Point Theorem.
Set $f(x) := x - \frac{g(x)}{g'(x)}$. Then given $x_0 \in I$ and $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = f(x_n)$.
Moreover, $f(\tilde{x}) = \tilde{x}$ if and only if $g(\tilde{x}) = 0$.

Let us restrict our discussion to the computation of $\sqrt{3}$. The Banach space $X$ is the space of real numbers $\mathbb{R}$ and $g(x) = x^2 - 3$, so

$$f(x) = x - \frac{x^2 - 3}{2x} = \frac{1}{2}(x + \frac{3}{x})$$

on $[\sqrt{3}, \infty) \to [\sqrt{3}, \infty)$. Note that $[\sqrt{3}, \infty)$ is a closed set of $\mathbb{R}$ containing $\sqrt{3}$. For $x \geq 0$ we have $\frac{1}{2}(x + 3/x) \geq \sqrt{3x/x} = \sqrt{3}$. Compute $f'$ and note that a differentiable function $f : I \to \mathbb{R}$ with a bounded derivative, $|f'(x)|L$ for $x \in I$ is Lipschitz continuous with constant $L$.

$$f'(x) = \frac{1}{2}(1 - \frac{3}{x^2})$$

and note that it's range is contained in $[0, 1/2]$ for $x \geq \sqrt{3}$. Hence we have $L = 1/2$ and by Banach's Fixed Point Theorem $\frac{1}{2}(x_n + \frac{3}{x_n}) \to \sqrt{3}$. Let's pick $x_0 = 2$ and thus $x_1 = 7/4$ and so $|x_1 - x_0| = 1/4$. Furthermore, we have

$$|x_n - \sqrt{3}| \leq \frac{(1/2)^n}{1 - 1/2}|x_1 - x_0| = \frac{1}{2^n} \cdot 2 \cdot \frac{1}{4} = \frac{1}{2^{n+1}}.$$

Hence

$$|x_n - \sqrt{3}| \leq \frac{1}{2^{n+1}}.$$

For $n = 4$, we have $|x_n - \sqrt{3}| \leq 1/1024 < 0.001$.

**Integral equations**

Equations of the following type appear naturally in mathematics, physics and engineering: Given functions $f : [a, b] \to \mathbb{R}$ and $k : [a, b] \times [a, b] \to \mathbb{R}$, a parameter $\lambda$, where $[a, b]$ denotes a finite interval of $\mathbb{R}$. Solve the **integral equation**

$$f(x) = \lambda \int_a^b k(x, y)f(y)dy + g(x)$$

for $g$. We will restrict our discussion to continuous functions $f$ and $k$. Note that the mapping

$$T(f)(x) = \int_a^b k(x, y)f(y)dy$$

is a continuous analogue of matrix multiplication, where the function $k$ on the rectangle $[a, b] \times [a, b]$ is the continuous variant of a matrix $(a_{ij})$ and one often calls $T$ an **integral operator** and $k$ its **kernel**. The fixed point theorem of Banach allows us to solve this integral equation for sufficiently small $\lambda$.

Note that $T : C[a,b] \to C[a,b]$ respects the vector space structure of $C[a,b]$: For any $\alpha, \beta \in \mathbb{R}$ and $f_1, f_2 \in C[a,b]$ we have

$$T(\alpha f_1 + \beta f_2) = \alpha T(f_1) + \beta T(f_2).$$

LEMMA 3.17. *Let* $f \in C[a,b]$ *and* $k \in C\Big([a,b] \times [a,b]\Big)$. *Then* $Tf \in C[a,b]$.

PROOF. For each fixed $x$ the function $K(x,y)$ is a continuous function of $y$ on $[a,b]$. Hence $K(x,y)f(y)$ is a continuous function of $y$ and so the integral in the definition of $T$ makes sense. **Claim:** For $f \in C[a,b]$ we also have $Tf \in C[a,b]$.

As a preparation we look at $|T(f)(x_1) - T(f)(x_2)|$ for $x_1 \neq x_2$:

$$\begin{aligned} |T(f)(x_1) - T(f)(x_2)| &\leq \Big| \int_a^b \Big(k(x_1,y) - k(x_2,y)\Big) f(y)dy \Big| \\ &\leq \int_a^b |k(x_1,y) - k(x_2,y)|\,|f(y)|dy. \end{aligned}$$

Since $k$ is continuous on $[a,b] \times [a,b]$, we have that $k$ is bounded on $[a,b] \times [a,b]$: $\|k\|_\infty \leq \|k\|_\infty$. We also have more control over $k$ as one would have for a continuous function. Namely, it is uniformly continuous on $[a,b] \times [a,b]$: For any $\delta > 0$ so that $|x_1 - x_2| < \delta$ we have

$$|k(x_1,y) - k(x_2,y)| \leq \varepsilon / \|f\|_\infty (b-a) \quad \text{for all } y \in [a,b].$$

Using this estimate we obtain that for $|x_1 - x_2| < \delta$

$$|T(f)(x_1) - T(f)(x_2)| \leq \varepsilon \quad \text{for all } y \in [a,b].$$

Hence $Tf$ is continuous on $[a,b]$. □

Furthermore $T$ is also compatible with the norm structure on $C[a,b]$, which follows from the estimates in the preceding proof:

$$\|T(f_1) - T(f_2)\|_\infty \leq \|k\|_\infty (b-a) \|f_1 - f_2\|_\infty.$$

Hence we are in the position to specify when $T_\lambda f(x) = g(x) + \lambda \int_a^b \mathrm{k(x,y)f(y)dy}$ is a contraction on $C[a,b]$: Namely, when $|\lambda| < \frac{1}{\|k\|_\infty (b-a)}$.

PROPOSITION 3.3.5. *Suppose* $g \in C[a,b]$ *and* $k \in C\Big([a,b] \times [a,b]\Big)$. *Then*

$$f(x) = \lambda \int_a^b k(x,y) f(y)dy + g(x)$$

*has a unique continuous solution* $\tilde{f}$ *on* $[a,b]$ *for* $|\lambda| < 1/\|k\|_\infty (b-a)$. *The solution can be found by iteration.*

PROOF. Consider the mapping $f(x) \mapsto T_\lambda f(x) := g(x)+\lambda \int_a^b k(x,y)f(y)dy$. For $f_1, f_2 \in C[a,b]$ we have

$$|T_\lambda f_1(x) - T_\lambda f_2(x)| = |g(x) - g(x)| + |\lambda| \int_a^b |k(x,y)||f_1(y) - f_2(y)|dy$$
$$\leq |\lambda| \int_a^b |k(x,y)||f_1(y) - f_2(y)|dy.$$

Since $k$ is bounded on $[a,b] \times [a,b]$ we have $|k(x,y)| \leq \|k\|_\infty$ for all $x, y \in [a,b]$:

$$|T_\lambda f_1(x)-T_\lambda f_2(x)| \leq |\lambda| \int_a^b |k(x,y)||f_1(y)-f_2(y)|dy \leq |\lambda|\|k\|_\infty \int_a^b |f_1(y)-f_2(y)|dy.$$

By the boundedness of $f_1 - f_2$ on $[a,b]$ we have that $|f_1(y) - f_2(y)| \leq \|f_1 - f_2\|_\infty$. Thus we have

$$|T_\lambda f_1(x)-T_\lambda f_2(x)| \leq |\lambda|\|k\|_\infty\|f\|_\infty \int_a^b 1dy = \lambda|(b-a)\|k\|_\infty\|f_1-f_2\|_\infty$$

Hence $T_\lambda$ is a contraction on the Banach space $(C[a,b], \|.\|_\infty)$ if $|\lambda|(b-a)\|k\|_\infty < 1$, i.e.

$$|\lambda| \leq ((b-a)\|k\|_\infty)^{-1}$$

and so Banach's fixed point theorem completes the argument.

□

Mappings of the form $T(f)(x) = \int_a^b k(x,y)f(y)dy$ are called **integral operators** and one may impose various conditions on $[a,b]$, the function $f$ and the kernel $k$ depending on your problem. We just point out that a specific choice of kernels gives integral operators with a one-dimensional range. Namely, if $k(x,y) = k_1(x)k_2(y)$, then

$$Tf(x) = \int_a^b k_1(x)k_2(y)f(y)dy = \langle k_2, f\rangle_2\, k_1(x),$$

is a scalar multiple of $k_1$. We denote functions of the form $k_1(x)k_2(y)$ by $(k_1 \otimes k_2)(x,y)$.