NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

# Examination paper for **TMA4145 Linear Methods**

**Academic contact during examination:** Franz Luef

**Phone:** 40614405

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**Permitted examination support material:** D:No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

**Other information:**
The exam consists of 12 problems, but the grade is based on your 10 best solutions. Although the problems are only formulated in English you can answer either in English or your favorite Norwegian language.

**Language:** English

**Number of pages:** ??

**Number of pages enclosed:** 0

**Checked by:**

| Informasjon om trykking av eksamensoppgave | |
|---|---|
| **Originalen er:** | |
| **1-sidig** ☐ | **2-sidig** ☐ |
| **sort/hvit** ☐ | **farger** ☐ |
| **skal ha flervalgskjema** ☐ | |

Date Signature

**Problem 1** Let $A$ be a non-empty subset of the real line $\mathbb{R}$.

**a)** Define the following notions: (a) the **infimum** of $A$; (b) the **supremum** of $A$; (3) the **closure** of $A$; (4) the **interior** of $A$; (5) the **boundary** of $A$.

**Solution:**

1. If $m$ is a lower bound of $A$ such that $m \geq m'$ for every lower bound $m'$, then $m$ is called the *infimum* of $A$, denoted by $m = \inf A$.
2. If $M$ is an upper bound of $A$ such that $M' \geq M$ for every upper bound $M'$, then $M$ is called the *supremum* of $A$, denoted by $M = \sup A$.
3. The *closure* of $A$, denoted by $\overline{A}$, is the intersection of all closed sets containg $A$.
4. The *interior* of a subset of $A$ of $\mathbb{R}$, denoted by int$A$, is the union of all open subsets of $\mathbb{R}$ contained in $A$.
5. The boundary of a subset $A$ of $\mathbb{R}$, denoted by bd$A$, is the set $\overline{A}\backslash\text{int}A$.

**b)** Assume that A is *bounded from above*. Show that the supremum of $A$ lies in the closure of $A$.

**Solution:**
Let $A \subset \mathbb{R}$ be bounded from above. By the axiom of completeness, the supremum of $A$ exists (as a real number), that is, $\sup A \in \mathbb{R}$.

Let $\varepsilon > 0$. Since $\sup A$ is the *least* upper bound of $A$, we have that $\sup A - \varepsilon$ cannot be an upper bound for $A$, so there is some element $a_\varepsilon \in A$ such that $a_\varepsilon > \sup A - \varepsilon$. Furthermore, since $\sup A$ is an upper bound of $A$, and since $a_\varepsilon \in A$, we must have that $a_\varepsilon \leq \sup A$. Thus

$$\sup A - \varepsilon < a_\varepsilon \leq \sup A\,.$$

For every $n \geq 1$ we may choose $\varepsilon = \frac{1}{n}$. Using the above, there is some element $a_n \in A$ such that

$$\sup A - \frac{1}{n} < a_n \leq \sup A\,.$$

We have obtained a sequence $(a_n) \subset A$ such that $\sup A - \frac{1}{n} < a_n \leq \sup A$ for all $n \geq 1$. Subtracting $\sup A$ from all sides of this inequality, we get that

$$-\frac{1}{n} < a_n - \sup A \leq 0\,,$$

which implies

$$|a_n - \sup A| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty, \quad \text{so } a_n \to \sup A\,.$$

**Problem 2** Consider the initial value problem:

$$\frac{dx}{dt} = f(t,x), \quad \text{and} \quad x(t_0) = x_0,$$

where $f$ is a function $f : U \times V \to \mathbb{R}$ defined on $U \times V$ of $\mathbb{R}^2$ such that $t_0$ lies in the interior of the interval $U$ and $x_0$ in the interior of the interval $V$, respectively.

**a)** Formulate the theorem of Picard-Lindelöf. Assume that $f$ is continuous in $t$ and uniformly Lipschitz in $x$:

$$|f(t,x) - f(t,x')| \leq L|x - x'| \quad \text{for all } t \in U, x, x' \in V.$$

Then the IVP has a unique local solution, i.e. there exists a $\delta > 0$ such that the IVP has a solution $x$ on $(x_0 - \delta, x_0 + \delta)$.

**b)** Solve the initial value problem

$$\frac{dx}{dt} = 2t(1 + x), \quad \text{and} \quad x(0) = 0,$$

by applying the theorem of Picard-Lindelöf. Compute the first three Picard iterations $x_1(t), x_2(t)$ and $x_3(t)$ starting from $x_0(t) = 0$.

**Solution:**
In this problem $f(x,t) = 2t(1 + x)$, which is continuous in $t$ and uniformly continuous on the closed interval $[-B, B]$ for any $B > 0$. Hence there exists a unique local solution. The formula for the Picard iteration is

$$x_{n+1}(t) = \int_0^t 2s(1 + x_n(s))ds = t^2 + \int_0^t 2s x_n(s)ds.$$

Since $x_0(t) = 0$ we have

$$x_1(t) = t^2, \quad x_2(t) = t^2 + \frac{t^4}{2!}, \quad x_3(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!}.$$

Note that $x_n(t)$ is the Taylor expansion of $e^{-t^2} - 1$. Hence $x_n(t) \to e^{-t^2} - 1$ and thus the solution actually exists for all $t \in \mathbb{R}$.

**Problem 3** Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

**a)** Compute the singular value decomposition of $A$.

**Solution:**

1. $A^*A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}$ has as characteristic polynomial $x^2 - 18x + 17 = (x - 17)(x - 1)$. Hence the eigenvalues of $A^*A$ are $\lambda_1 = 17$ and $\lambda_2 = 1$. The corresponding normalized eigenvectors are $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Consequently, we have

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

2. The singular values of $A$ are $\sigma_1 = \sqrt{17}$ and $\sigma_2 = 1$. Thus

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3. The first two columns of $U$ are given by

$$u_1 = \frac{1}{\sqrt{17}} \frac{1}{\sqrt{2}} A v_1 = \frac{1}{\sqrt{34}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$$

and by

$$u_2 = \frac{1}{1} A v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Consequently, $U$ has the form

$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & x_1 \\ \frac{4}{\sqrt{34}} & 0 & x_2 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & x_3 \end{pmatrix}$$

The last column is determined by the assumption that it has to be orthogonal to the first two columns. The choice

$$u_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

satisfies these conditions, but there are many other ways to complete the first two columns to become an orthonormal basis for $\mathbb{C}^3$,

$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix}.$$

4. The SVD of $A$ is

$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

**b)** Use the result of a) to find:
(1) Bases for the following vector spaces: $\ker(A), \ker(A^*), \operatorname{ran}(A), \operatorname{ran}(A^*)$.

**Solution:**

$$\ker(A^*) = \{u_3\}, \ker(A) = \{0\}, \operatorname{ran}(A^*) = \{v_1, v_2\}, \operatorname{ran}(A) = \{u_1, u_2\}.$$

(3) The pseudo-inverse of $A$.

**Solution:**

$$A^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{4}{\sqrt{34}} & \frac{3}{\sqrt{34}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{17}} & \frac{-3}{\sqrt{17}} & \frac{2}{\sqrt{17}} \end{pmatrix}.$$

**Problem 4** Let $\|.\|_a$ and $\|.\|_b$ be two norms on a vector space $X$.

**a)** Show that $\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ is a norm on $X$. Furthermore, show if a sequence $(x_n)$ converges in $(X, \|.\|)$, then it converges in $(X, \|.\|_a)$ and in $(X, \|.\|_b)$.

**Solution:**
$\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ is a norm, because

1. $\|x\| = 0$ if and only if $\|x\|_a = 0$ and $\|x\|_b = 0$, which is the case only if $x = 0$.
2. $\|\lambda x\| = (\|\lambda x\|_a^2 + \|\|ax\|_b^2)^{1/2} = (\lambda^2\|x\|_a^2 + \lambda^2\|x\|_b^2)^{1/2} = |\lambda|(\|x\|_a^2 + \|x\|_b^2)^{1/2} = |\lambda|\|x\|$.
3. Let us compute $\|x+y\|^2$ and $(\|x\| + \|y\|)^2$.

$$\begin{aligned}(\|x\| + \|y\|)^2 &= (\|x\|_a + \|x\|_b)^2 + (\|y\|_a + \|y\|_b)^2 \\ &= \|x\|_a^2 + \|x\|_b^2 + \|y\|_a^2 + \|y\|_b^2 + \\ &2\|x\|_a\|y\|_a + 2\|y\|_a\|x\|_b + 2\|x\|_a\|y\|_b + 2\|x\|_b\|y\|_b\end{aligned}$$

$$\begin{aligned}\|x+y\|^2 &= \|x+y\|_a^2 + \|x+y\|_b^2 \\ &\leq (\|x\|_a + \|y\|_a)^2 + (\|x\|_b + \|y\|_b)^2 = \|x\|_a^2 + \|x\|_b^2 + \|y\|_a^2 + \|y\|_b^2 + \\ &2\|x\|_a\|x\|_b + 2\|y\|_a\|y\|_b,\end{aligned}$$

which gives us

$$(\|x\| + \|y\|)^2 - \|x+y\|^2 = 2\|y\|_a\|x\|_b + 2\|x\|_a\|y\|_b \geq 0$$

and consequently

$$\|x+y\| \leq \|x\| + \|y\|.$$

For the convergence statement, we just observe that $\|x\|_a \leq \|x\|$ and $\|x\|_a \leq \|x\|$, which yields the desired assertions.

**b)** Suppose there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b$$

holds for all $x \in X$, i.e. $\|.\|_a$ and $\|.\|_b$ are equivalent norms on $X$.

Show that there exist constants $C_1', C_2' > 0$ such that

$$C_1'\|x\|_a \leq \|x\|_b \leq C_2'\|x\|_a$$

holds for all $x \in X$.

**Solution:**

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b$$

implies that $C_1\|x\|_b \leq \|x\|_a$, hence

$$\|x\|_b \leq C_1^{-1}\|x\|_a,$$

and from $\|x\|_a \leq C_2\|x\|_b$ we deduce

$$C_2^{-1}\|x\|_a \leq \|x\|_b$$

. Consequently, we have

$$C_2^{-1}\|x\|_a \leq \|x\|_b \leq C_1^{-1}\|x\|_a,$$

so $C_1' = C_2^{-1}, C_2' = C_1^{-1}$.

Determine the constants $C_1$ and $C_2$ for the sup-norm $\|.\|_\infty$ and $\|.\|_p$-norm, $1 \leq p < \infty$. on $\mathbb{R}^n$:

$$C_1\|x\|_\infty \leq \|x\|_p \leq C_2\|x\|_\infty.$$

We have $C_1 = 1$ and $C_2 = n^{1/p}$, because $\max\{|x_i| : i = 1, ..., n\} = \max\{|x_i|^p : i = 1, ..., n\}^{1/p} \leq (\sum_{i=1}^n |x_i|^p)^{1/p}$ and $(\sum_{i=1}^n |x_i|^p)^{1/p} \leq (\sum_{i=1}^n \max\{|x_i| : i = 1, ..., n\}^p)^{1/p} \leq n^{1/p} \max\{|x_i| : i = 1, ..., n\}$.

**Problem 5** Let $M$ be the subspace of $\ell^2$ defined by

$$M = \{x = (x_k)_{k\in\mathbb{N}} \in \ell^2 : x_{2k} = 0 \text{ for } k = 1, 2, ...\}.$$

**a)** Show that $M$ is a closed subspace of $\ell^2$ and determine its orthogonal complement $M^\perp$.

**Solution:**
Suppose $(x^n)_{n\in\mathbb{N}}$, where $x^n = (x_k^n)_{k\in\mathbb{N}}$, is a sequence in $M$ converging to $x = (x_k)_{k\in\mathbb{N}}$ in $\ell^2$. Since $x_{2j}^n = 0$ for $j = 1, 2, ...$ we have

$$|x_{2j}| = |x_{2j} - x_{2j}^n| = (|x_{2j} - x_{2j}^n|^2)^{1/2} \leq (\sum_{j\in\mathbb{N}} |x_j - x_j^n|^2)^{1/2} = \|x - x^k\|_2$$

for all $j, j \in \mathbb{N}$. Hence in the limit as $k \to \infty$ we get that $|x_{2j}| = 0$ for all $j \in \mathbb{N}$. Thus $x \in M$ and M is a closed subspace of $\ell^2$.

The natural candidate for the orthogonal complement of $M$ is the subspace

$$N = \{(x_k)_{k\in\mathbb{N}} | \, x_{2j-1} = 0, \quad j = 1, 2, ...\}.$$

Now, for $x \in M$ and $y \in N$ we have $\langle x, y\rangle = 0$. Hence $N \subseteq M^\perp$.
Suppose $y \in M^\perp$. Then $\langle x, y\rangle = 0$ for all $x \in M$. Let us take the standard basis $\{e_k : k \in \mathbb{N}\}$. Then we $e_{2j-1} \in M$ for all j and we have for $x = e_{2j-1}$ that $\langle x, e_{2j-1}\rangle = x_{2j-1} = 0$. Consequently, $y \in N$ and thus $M^\perp = N$.

**b)** Determine the orthogonal projection $P$ from $\ell^2$ onto $M$ without using the projection theorem and show that $P = P^*$ and its operator norm $\|P\| = 1$.

**Solution:**
The orthogonal projection $Px$ of $x$ onto $M$ is the best approximation of x in $M$ such that its error is in the orthogonal complement. Hence $x - Px \in M^\perp$, i.e $\langle x - Px, y \rangle = 0$ for all $y \in M$. Hence $x - Px = (x_{2j-1})$, so $Px = (x_{2j}) = (x_2, x_4, x_6, ...)$. We also have that $M \cap M^\perp = \{0\}$ since $(x_1, x_2, x_3, ...) = (x_2, x_4, ...) + (x_1, x_3, ...)$ for all $x \in \ell^2$. Hence $P^2 = P$ and

$$\langle Px, y \rangle = \sum_{k=1}^{\infty} x_2 k \overline{y}_{2k} = \langle x, Py \rangle,$$

i.e. $P = P^*$.
By Pythagoras we have $\|x\|_2^2 = \|Px\|_2^2 + \|y\|_2^2$ where $y \in M^\perp$. Thus $\|Px\| \leq \|x\|$. On the other hand there exists an $x \in \ell^2$ such that $Px \neq 0$, but $P(Px) = Px$. Thus $\|P(Px)\| = \|Px\|$, so we have $\|P\| \geq 1$. Hence $\|P\| = 1$.
Now for $x \in \ell^2$ to $M$.

**Problem 6** Let $X$ be a separable Hilbert space and $\{e_k : k = 0, 1, 2, ...\}$ an orthonormal basis for $X$. We define the linear operator $S$ by $S(e_k) = e_{k+1}$ for $k = 0, 1, 2, ...$ .

**a)** Suppose $a = (a_0, a_1, ...) \in \ell^2$ is the coefficient sequence of $x \in X$:

$$x = \sum_{k=0}^{\infty} a_k e_k.$$

Describe the operator $S$ in terms of the coefficient sequence $(a_0, a_1, ...)$, i.e. as an operator on $\ell^2$. Determine $S^*$ on $\ell^2$ and find $S^*(e_k)$ for $k = 0, 1, ....$ Compute the operator norm of $S$.

**Solution:**
By definition we have

$$S(x) = \sum_{k=0}^{\infty} a_k S(e_k) = \sum_{k=0}^{\infty} a_k e_{k+1}.$$

In order to get the action of $S$ on $\ell^2$ we have to change the summation index:

$$\sum_{k=0}^{\infty} a_k S(e_k) = \sum_{k=1}^{\infty} a_{k-1} e_k = 0e_0 + a_0 e_1 + a_1 e_2 + \cdots$$

.

Hence for $a = (a_0, a_1, ...)$ we have

$$S(a_0, a_1, ...) = (0, a_0, a_1, a_2, ...)$$

is a shift operator. Then we have that the adjoint of $S$ is defined by

$$\langle Sa, b\rangle = \sum_{k=0}^{\infty} a_{k-1}\bar{b}_k = \sum_{k=0}^{\infty} a_k \bar{b}_{k+1} = \langle a, S^* b\rangle.$$

Hence $S^*$ is the other shift operator on $\ell^2$

$$S^*(a_0, a_1, ...) = (a_1, ...),$$

and in terms of the basis elements $S^*$ is defined by $S^* e_0 = 0$ and $S^* e_k = e_{k-1}$ for $k = 1, 2, ...$ .The operator norm of $S$ equals

$$\|S\| = \sup\{\|Sa\|_2 : \ a \in \ell^2 \ \text{ with } \ \|a\|_2 = 1\}.$$

Since $\|Sa\|_2^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots = \|a\|_2^2$ we have that $\|S\| = 1$.

**b)** Determine if $S$ and $S^*$ are injective and/or surjective, respectively. Determine $S^*S$ and $SS^*$, their kernels and ranges, respectively.

**Solution:**
We consider $S$ and $S^*$ as linear operators on $\ell^2$. $S^*$ is not injective, $(1, 0, 0, ...)$ gets mapped to $(0, 0, ...)$, and it is surjective: any $a \in \ell^2$ lies in the range of $S^*$: $S^*(0, a_0, a_1, ...) = (a_0, a_1, ...)$. The map $S$ is injective and not surjective: $Sa = (0, a_0, a_1, ...) = 0$ if and only if $a = 0$; and $(1, 0, 0, ..)$ does not lie in the range of $S$.

$$S^*Sa = (a_0, a_1, a_2, ...) \ \text{ and } \ SS^* = (0, a_1, a_2, ...).$$

Hence the kernel of $SS^*$ is $\{(\alpha, 0, 0, ...) : \ \alpha \in \mathbb{C}\}$ and the range of $SS^*$ is $\ell^2$. Since $S^*S = I$ its kernel is just $\{0\}$ and its range is $\ell^2$.