Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2016

Exercise set 11

Justify all the steps in your answers.

$\boxed{1}$ Let U be a linear transformation on a finite-dimensional innerproduct space X. Show that the following statements are equivalent.

1. U is unitary, i.e. $U^*U = I = UU^*$.
2. The columns of U are an orthonormal basis of X.
3. The rows of U are an orthonormal basis of X.

$\boxed{2}$ Let X be a finite-dimensional vector space and X' its dual space, i.e. the space of all linear mappings from X to $\mathbb{C}$.

a) Show that if $\varphi(x) = 0$ for all $\varphi \in X'$, then $x = 0$.

b) Suppose $\{x_1, ..., x_n\}$ is a basis of X. Then we define a linear functional x_i^* on X that maps x_i to 1 and all the other basis vectors to 0, i.e. $x_i^*(x_j) = \delta_{ij}$. Show that for all $x \in X$ we have
$$x = \sum_{i=1}^{n} x_i^*(x) x_i.$$

c) Show that $\{x_1^*, ..., x_n^*\}$ is a basis for X', the dual basis of $\{x_1, ..., x_n\}$.

d) Formulate the previous two statements for a a finite-dimensional innerproduct space $(X, \langle .,. \rangle)$. (Can you describe the linear functionals in terms of the innerproduct?) s of the innerproduct?)

$\boxed{3}$ Let $\mathcal{P}_2$ be the space of polynomials of degree at most n.

a) Given different numbers x_0, x_1, x_2. Determine polynomials $\{l_i : i = 0, ..., 2\}$ such that
$$l_i(t_j) = \delta_{ij}$$
and show that
$$f(x) = \sum_{j=0}^{2} y_j l_j(x)$$
is a polynomial with values $f(x_j) = y_j$. In other words, given three tuples $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, then one can find a polynomial that has at the points x_0, x_1, x_2 the values y_0, y_1, y_2.

b) Can you link the results from Problem 2 with this statement? Formulate the problem for polynomials of degree at most n.

4 Let X be finite-dimensional innerproduct space. Suppose $\{x_1, ..., x_n\}$ is a set in X. We define the coefficient operator $C : X \to \mathbb{C}^n$ by

$$x \mapsto (\langle x, x_i \rangle)_{i=1}^n,$$

and the synthesis operator $D : \mathbb{C}^n \to X$ by

$$(\alpha_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_i x_i.$$

a) Show that $C^* = D$. Determine $S = C^*C$ and $G = CC^*$.

b) Consider the following set of vectors in $\mathbb{C}^3$:

$$\{\sqrt{2}3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{2}3 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \sqrt{2}3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \}$$

Determine S and G for this set. What are the rank and the kernel of S (depending on the linear independence/dependence and spanning properties of the set)?

5 Let P_3 be the space of polynomials of degree at most 3.

a) Show that $\{B_0^3(x) = (1-x)^3, B_1^3(x) = 3x(1-x)^2, B_2^3(x) = 3x^2(1-x), B_3^3(x) = x^3\}$ is a basis for $\mathcal{P}^3$, known as the Bernstein basis. Try to define a Bernstein basis for the space of polynomials of degree at most n.

b) Since $\{B_i^3(x) : i = 0, ..., 3\}$ is a basis of $\mathcal{P}_3$ there exist unique coefficients $\alpha_0, ..., \alpha_3$ for any $f \in \mathcal{P}_3$ such that

$$f(x) = \alpha_0 B_0^3(x) + \alpha_1 B_1^3(x) + \alpha_2 B_2^3(x) + \alpha_3 B_3^3(x).$$

On the other hand we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Express α_i in terms of a_i for $i = 0, ..., 3$. In other words, how does one convert a polynomial in monomial form to one in the Bernstein basis?