Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2016

Exercise set 3

Justify all the steps in your answers.

1 Let V be a vector space and let $S \subset V$ be a non-empty subset.

a) Show that S is a basis in V if and only if every vector $v \in V$ has a unique representation as a linear combination $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$ with vectors $v_1, \ldots, v_n \in S$.

b) Consider the vector space $\mathbf{P}_2$ of polynomials of degree ≤ 2.
For each of the following sets, determine if it is linearly independent in $\mathbf{P}_2$, if it spans $\mathbf{P}_2$ and if it is a basis in $\mathbf{P}_2$:

(i) $\{1 - x, 1 + x, x^2\}$.
(ii) $\{1 + x, 1 + x^2, x - x^2\}$.

2 Consider the vector space $\mathbf{M}_2$ of all 2×2 matrices with real entries. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

a) Find span $\{A_1, A_2\}$.

b) Does the set $\{A_1, A_2, A_3, A_4\}$ span $\mathbf{M}_2$? Is this set linearly independent? Is this set a basis in $\mathbf{M}_2$?

3 Let $\mathbf{P}_4$ be the vector space of real polynomials of degree at most 4.

a) Show that the sets $U, V \subset \mathbf{P}_4$ defined by

$$U := \{p \in \mathbf{P}_4 : \ p(-1) = p(1) = 0\},$$
$$V := \{p \in \mathbf{P}_4 : \ p(1) = p(2) = p(3) = 0\}$$

are subspaces of $\mathbf{P}_4$.

b) Determine the subspace $U \cap V$.

c) Describe bases for U, V and $U \cap V$.

4 Consider the vector space $\mathbb{R}^n$ and let $1 \leq p \leq \infty$.
Recall that the function $\|\cdot\|_p \colon \mathbb{R}^n \to \mathbb{R}$ is defined, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, by

$$\|x\|_p := (|x_1|^p + \ldots + |x_n|^p)^{1/p} \quad \text{if } 1 \leq p < \infty,$$
$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}.$$

This problem asks that you complete the remaining steps in proving that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

a) For all $1 \leq p \leq \infty$, prove the positivity axiom of the norm.

b) For all $1 \leq p \leq \infty$, prove the homogeneity axiom of the norm.

c) For $p = 1$ and $p = \infty$ prove the triangle inequality axiom of the norm.

5 **a)** Draw the unit balls of $(\mathbb{R}^2, \|\cdot\|_1)$, $(\mathbb{R}^2, \|\cdot\|_2)$ and of $(\mathbb{R}^2, \|\cdot\|_\infty)$.

b) Let $p \geq 1$. Prove that every ball (of any center and radius) of $(\mathbb{R}^n, \|\cdot\|_p)$ contains a ball of $(\mathbb{R}^n, \|\cdot\|_\infty)$; vice-versa, prove that every ball of $(\mathbb{R}^n, \|\cdot\|_\infty)$ contains a ball of $(\mathbb{R}^n, \|\cdot\|_p)$.