Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2015

Exercise set 13

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Suppose that $0 < a < b$ are real numbers and let $k\colon [a,b]^2 \to \mathbb{C}$ be a continuous function on the square $[a,b]^2 = [a,b] \times [a,b]$. Then

$$K = \sup_{(x,y)\in[a,b]^2} |k(x,y)| < \infty$$

by compactness of $[a,b]^2$.

Write X for $C([a,b]) = C([a,b],\mathbb{C})$ with the usual sup-norm $\|\cdot\|_\infty$.

a) Show that the (linear) integral operator $T_k\colon X \to X$ defined by

$$(T_k f)(x) = \int_a^b k(x,y) f(y)\, dy$$

is bounded.

b) The function k is (confusingly) known as the kernel of the integral operator T_k. Find a kernel $h\colon [a,b]^2 \to \mathbb{C}$ such that $T_k^2 = T_h$.

c) Consider T_k as an operator on the Hilbert space $L^2([a,b])$. Determine its adjoint operator, T_k^*.

2 Suppose that $A = (a_{ij})_{i,j\in\mathbb{N}}$ is an infinite matrix of complex numbers, with

$$C = \sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty.$$

Show that the linear operator $T\colon \ell^2 \to \ell^2$ defined by matrix multiplication with A,

$$T(x_i)_{i\in\mathbb{N}} = (y_i)_{i\in\mathbb{N}}, \quad \text{where} \quad y_i = \sum_{j=1}^{\infty} a_{ij} x_j,$$

is bounded.

3 Compute the exponential of the following operators:

a)

$$D = \operatorname{diag}(d_1, \dots, d_n) \quad \text{(Diagonal matrix)}$$

b)

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{(Counterclockwise rotation by } 90^\circ \text{ on } \mathbb{R}^2.\text{)}$$

c) A nilpotent operator $N \in B(X)$, where X is a Banach space.

4 Find the singular value decomposition of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

5 Consider the Banach space ℓ^∞ (with the usual norm). Let $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty$ and define the (linear) multiplication operator $M_\lambda \colon \ell^\infty \to \ell^\infty$ by

$$M_\lambda (x_n)_{n\in\mathbb{N}} = (\lambda_n x_n)_{n\in\mathbb{N}}$$

a) Show that M_λ is injective if and only if $\lambda_n \neq 0$ for all $n \in \mathbb{N}$.

b) Show that M_λ is surjective if and only if $\delta = \inf_{n\in\mathbb{N}} |\lambda_n| > 0$.

c) Show that M_λ does not have closed range if $\lambda = (1/n)_{n\in\mathbb{N}}$.