Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 9

Note that there are several ways to approach the problems, but that only one is presented here. Other solutions may be just as valid!

1 **a)** The equation is equivalent to the fixed-point equation

$$f(x) = x \tag{1}$$

for the function $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ defined in the hint. We need to show that this function is a contraction. Now, for $x, y \in \mathbb{R}^4$ we have

$$d_\infty(f(x), f(y)) = \max_k |((Ax + b) - (Ay + b))_k| = \max_k |(A(x - y))_k|,$$

and for $z \in \mathbb{R}^4$ we have

$$\begin{aligned}
(Az)_1 &= \frac{1}{3}z_2 - \frac{1}{3}z_3 + \frac{1}{6}z_4 \\
(Az)_2 &= \frac{1}{4}z_1 + \frac{2}{5}z_3 - \frac{1}{5}z_4 \\
(Az)_3 &= \frac{1}{4}z_1 + \frac{1}{4}z_2 - \frac{1}{4}z_4 \\
(Az)_4 &= \frac{1}{3}z_2 - \frac{1}{3}z_3.
\end{aligned}$$

Observe that

$$|(Az)_1| \le \frac{1}{3}|z_2| + \frac{1}{3}|z_3| + \frac{1}{6}|z_4| \le \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{6}\right) \max_k |z_k| = \frac{5}{6} \max_k |z_k|,$$

and similarly

$$|(Az)_2| \le \frac{17}{20} \max_k |z_k|, \quad |(Az)_3| \le \frac{3}{4} \max_k |z_k|, \quad |(Az)_4| \le \frac{2}{3} \max_k |z_k|.$$

In particular

$$\max_k |(Az)_k| \le \left(\max\left\{\frac{5}{6}, \frac{17}{20}, \frac{3}{4}, \frac{2}{3}\right\}\right) \max_k |z_k| = \frac{17}{20} \max_k |z_k|,$$

and so

$$d_\infty(f(x), f(y)) = \max_k |(A(x - y))_k| \le \frac{17}{20} \max_k |(x - y)_k| = \frac{17}{20} d_\infty(x, y)$$

shows that f is a contraction on $(\mathbb{R}^4, d_\infty)$.

Since f is a contraction on the complete metric space $(\mathbb{R}^4, d_\infty)$, we can use the Banach fixed-point theorem to conclude that (1) has a unique solution x^*, and that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n = f(x_{n-1}) = Ax_{n-1} + b$$

for $n \geq 1$ converges to x^* in $(\mathbb{R}^4, d_\infty)$ for any $x_0 \in \mathbb{R}^4$.

b) We find

$$x_1 = f(x_0) = Ax_0 + b = b$$

when $x_0 = 0$. Continuing, the second iteration is

$$x_2 = f(x_1) = Ax_1 + b = Ab + b = \begin{bmatrix} 1 & 1/3 & -1/3 & 1/6 \\ 1/5 & 1 & 2/5 & -1/5 \\ 1/4 & 1/4 & 1 & -1/4 \\ 0 & 1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 18/5 \\ 7/2 \\ -2 \end{bmatrix}.$$

(The distance between x_2 and x^* in $(\mathbb{R}^4, d_\infty)$ is approximately 0.7.)

2 **a)** For $x, y \in C([0,1])$, we have

$$\begin{aligned} |(Gx)(t) - (Gy)(t)| &= \left| \int_0^t sx(s)\,ds - \int_0^t sy(s)\,ds \right| \\ &= \left| \int_0^t s(x(s) - y(s))\,ds \right| \\ &\leq \int_0^t s|x(s) - y(s)|\,ds \\ &\leq \int_0^t sd_\infty(x,y)\,ds \\ &= \frac{1}{2} d_\infty(x,y) \end{aligned}$$

for every $t \in [0,1]$. Hence,

$$d_\infty(Gx, Gy) \leq \frac{1}{2} d_\infty(x,y), \qquad x, y \in C([0,1]),$$

and so G is a contraction on $(C([0,1]), d_\infty)$.

b) The formula is correct for $n = 1$, because

$$(F(x_0))(t) = (F(0))(t) = \frac{t^2}{2} - (G0)(t) = \frac{t^2}{2} = \sum_{k=1}^{1} (-1)^{k+1} \frac{t^{2k}}{2^k k!}.$$

Suppose therefore that it holds for $n = m$. Then

$$\begin{aligned} (F^{m+1}(x_0))(t) = (F(F^m(x_0)))(t) &= \frac{t^2}{2} - (GF^m(x_0))(t) \\ &= \frac{t^2}{2} - \int_0^t s \sum_{k=1}^{n} m(-1)^{k+1} \frac{s^{2k}}{2^k k!}\,ds \\ &= \frac{t^2}{2} - \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^k k!} \int_0^t s^{2k+1}\,ds. \end{aligned}$$

If we now compute the integral and make the substitution $l = k+1$, then we find

$$\begin{aligned}(F^{m+1}(x_0))(t) &= \frac{t^2}{2} - \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^k k!}\frac{t^{2k+2}}{2k+2} \\ &= \frac{t^2}{2} + \sum_{k=1}^{m}(-1)^{(k+1)+1}\frac{1}{2^{k+1}(k+1)!}t^{2(k+1)} \\ &= \frac{t^2}{2} + \sum_{l=2}^{m+1}(-1)^{l+1}\frac{1}{2^l l!}t^{2l} \\ &= \sum_{l=1}^{m+1}(-1)^{l+1}\frac{t^{2l}}{2^l l!},\end{aligned}$$

which shows that the formula holds for $n = m+1$. Hence it holds for all $n \in \mathbb{N}$.

c) For $x, y \in C([0,1])$ we have

$$d_\infty(F(x), F(y)) = d_\infty(Gx, Gy) \le \frac{1}{2} d_\infty(x,y),$$

so F is also a contraction on $(C([0,1]), d_\infty)$. As this is a complete metric space, the Banach fixed-point theorem says that F has a unique fixed point x^* in $C([0,1])$. It also says that this fixed point can be found as the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = F^n(x_0)$ for any $x_0 \in C([0,1])$. In particular, if $x_0 = 0$ we get

$$x_n(t) = \sum_{k=1}^{n}(-1)^{k+1}\frac{t^{2k}}{2^k k!}, \qquad t \in [0,1], n \in \mathbb{N}$$

by **b)**. Since x_n converges uniformly, and therefore pointwise, to x^*, we must have

$$x^*(t) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k}}{2^k k!} = 1 - \sum_{k=0}^{\infty}\frac{(-t^2/2)^k}{k!} = 1 - e^{-t^2/2}$$

for $t \in [0,1]$.

3 By using the definition of the metric, we find

$$\begin{aligned} d_1(Ax, Ay) &= \sum_{k=1}^{n} |(Ax)_k - (Ay)_k| = \sum_{k=1}^{n} |(A(x-y))_k| \\ &= \sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ik}(x-y)_k\right| \\ &\le \sum_{k=1}^{n}\sum_{i=1}^{n} |a_{ik}||x_k - y_k| = \sum_{k=1}^{n}\left(\sum_{i=1}^{n}|a_{ik}|\right)|x_k - y_k| \\ &\le \sum_{k=1}^{n}\left(\max_j \sum_{i=1}^{n}|a_{ij}|\right)|x_k - y_k| \\ &= \left(\max_j \sum_{i=1}|a_{ij}|\right) d_1(x,y).\end{aligned}$$

The result follows, because the expression in the parentheses is smaller than one by assumption.

4 **a)** The equation is of the form

$$\dot{y}(t) = f(t, y(t))$$

where the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t, y) = t + y^2.$$

The function f is clearly continuous, and it is locally Lipschitz in the second variable because

$$|f(t,y) - f(t,x)| = |(t + y^2) - (t + x^2)| = |y^2 - x^2| = |y + x||y - x|$$

for all $t, x, y \in \mathbb{R}$. Specifically, if $M > 0$ then

$$|f(t,y) - f(t,x)| = |y + x||y - x| \le (|x| + |y|)|y - x| \le 2M|y - x|$$

for all $t \in \mathbb{R}$ and $x, y \in [-M, M]$.

(This implies the condition used in the lecture notes: If $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$, then we can choose M large enough so that (t_0, y_0) is an interior point of $\mathbb{R} \times [-M, M]$. Finally, choose $\epsilon > 0$ such that the ball $B_\epsilon(t_0, y_0)$ is contained in the rectangle. Then f is Lipschitz in y in this ball, with constant $L = 2M$.)

The Picard–Lindelöf theorem therefore applies, and yields *local* existence and uniqueness of a solution to the initial value problem.

(Note that this solution cannot be written down in terms of elementary functions.)

b) We find

$$\begin{aligned}
y_0(t) &= 0, \\
y_1(t) &= \int_0^t f(s, y_0(s))\, ds = \int_0^t s\, ds = \frac{1}{2}t^2, \\
y_2(t) &= \int_0^t f(s, y_1(s))\, ds = \int_0^t (s + (s^2/2)^2)\, ds = \frac{1}{2}t^2 + \frac{1}{20}t^5
\end{aligned}$$

for $t \in \mathbb{R}$. Note that the Picard–Lindelöf theorem only guarantees that these iterations converge to a solution of the initial value problem in a small interval around 0.

(In fact, it is possible to prove that the solution blows up in finite time. Numerical evidence seems to suggest that the maximal existence interval is $(-\infty, 1.986\ldots)$. See Figure 1 below.)

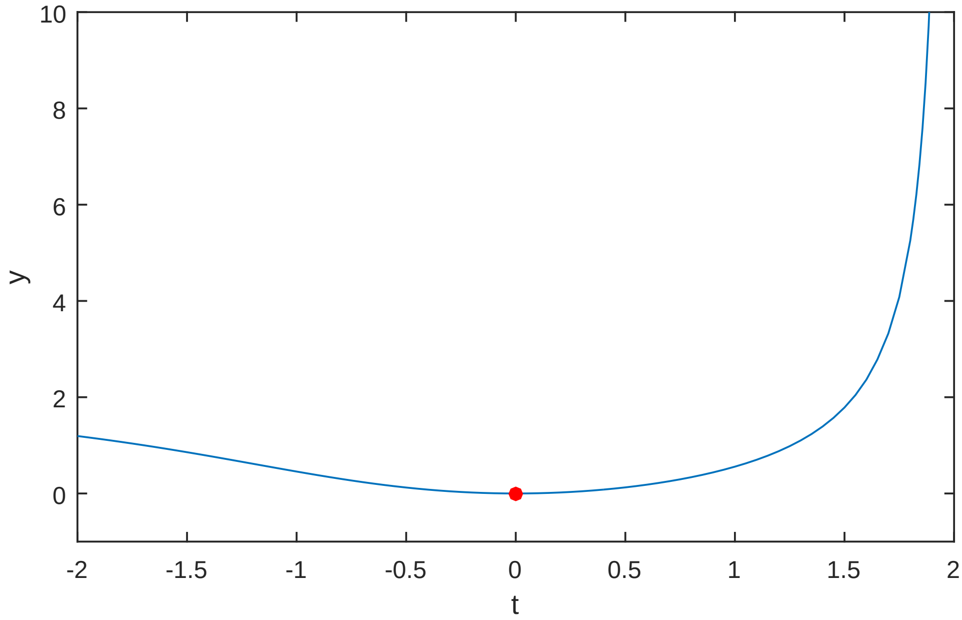

Figure 1: Numerical solution of the initial value problem.