Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2015

Exercise set 6

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let T be the cyclic (left) shift on $\mathbb{R}^3$. That is, the linear operator $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_2, x_3, x_1).$$

If $p \in \mathcal{P}_2$, say $p(t) = a_0 + a_1 t + a_2 t^2$, we can define a new operator

$$p(T) = a_0 I + a_1 T + a_2 T^2. \tag{1}$$

Let

$$\mathcal{C} = \{p(T) : p \in \mathcal{P}_2\}$$

be the collection of all such operators.

a) Describe the matrices representing operators in $\mathcal{C}$.

b) Show that the sum and product of two operators in $\mathcal{C}$ is again in $\mathcal{C}$, and that they commute.
(Two operators A, B commute if $AB = BA$; that is, their product does not depend on the order.)

c) In problem set 5, you determined the eigenvalues and eigenvectors of T (on $\mathbb{C}^3$). Use this to diagonalize all members of $\mathcal{C}$.

2 Let T be a nilpotent operator on a finite-dimensional vector space V. Show that $I - T$ is invertible, and give a formula for its inverse. Apply this result to the differentiation operator D on $\mathcal{P}_3$ (that is, $(Dp)(x) = p'(x)$).

3 Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}.$$

a) Compute the eigenvalues of A. What are their algebraic and geometric multiplicities?

b) Find a Jordan normal form PJP^{-1} of A.
(You do not have to explicitly invert the P you find.)