Exercise set 5

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Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let X be a vector space. We denote the annihilator

$$\{f \in X' : f(x) = 0 \text{ for all } x \in U\}$$

of a subspace $U \subseteq X$ by $U^\perp$. The annihilator $U^\perp$ is a subspace of the dual space X'. Let Y_1 and Y_2 be subspaces of X. Show that:

a) If $Y_1 \subseteq Y_2$, then $Y_2^\perp \subseteq Y_1^\perp$ (note the reversal!)

b) $Y_1^\perp + Y_2^\perp \subseteq (Y_1 \cap Y_2)^\perp$
Bonus: Show the other inclusion when X is finite-dimensional (it is not true in general).

c) $(Y_1 + Y_2)^\perp = Y_1^\perp \cap Y_2^\perp$

2 Let $\mathcal{P}_3$ be the space of polynomials of degree at most 3. For each $x \in \mathbb{R}$ we define the map $l_x \colon \mathcal{P}_3 \to \mathbb{R}$ by $l_x(p) = p(x)$; the evaluation of polynomials at x.

a) Show that l_x is a linear functional, and therefore a member of $\mathcal{P}_3'$.

b) Let x_1, x_2, x_3, x_4 be distinct real numbers. Show that $\{l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}\}$ is a basis for $\mathcal{P}_3'$.
Hint: Find a basis for $\mathcal{P}_3$ which this is the dual basis to.

c) Revisit Lagrange's interpolation formula from the point of view of **b)**.

3 We denote by $\mathcal{M}_n(\mathbb{R})$ the vector space of real n-by-n matrices.

a) Show that $\dim \mathcal{M}_n(\mathbb{R}) = n^2$.

b) Fix a nonzero $B \in \mathcal{M}_n(\mathbb{R})$ and define the map $T \colon \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ by

$$T(A) = BA - AB.$$

Show that $\operatorname{rank}(T) \leq n^2 - 1$ and $\operatorname{nullity}(T) \geq 1$.

4 Consider the matrices

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 0 & 0 & & \cdots & 0 \end{bmatrix}.$$

a) Describe the actions of T_1 and T_2 as linear mappings on $\mathbb{C}^n$.

b) Show that $T_1^n = I$, $T_2^{n-1} \neq 0$ and $T_2^n = 0$.

c) Find all eigenvectors and eigenvalues for T_1 and T_2.

5 Let P be a projection on a finite-dimensional vector space V and let q be a polynomial

$$q(t) = \sum_{j=0}^{n} a_j t^j.$$

Show that

$$q(P) = a_0 I + \left(\sum_{j=1}^{n} a_j \right) P$$

6 Let A and S be square matrices of the same size, with S invertible. Show that

$$p(S^{-1}AS) = S^{-1}p(A)S$$

for every polynomial p.

7 Consider the linear operator P defined on $\mathcal{M}_n(\mathbb{R})$ by

$$P(A) = \frac{1}{2}(A + A^T).$$

Show that P is a projection, and describe its image and kernel.