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TMA4145 Linear Methods
Fall 2015

Solutions to exercise set 4

Note that there are several ways to approach the problems, but that only one is presented here. Other solutions may be just as valid!

$\boxed{1}$ **a)** A point $w \in X$ lies in both U and V if and only if

$$w = (x, y, 0) = (\tilde{x}, 0, \tilde{z}) \tag{1}$$

for some choice of $x, y, \tilde{x}, \tilde{z} \in \mathbb{R}$. Equation (1) is satisfied if and only if $\tilde{x} = x$, $y = 0$ and $\tilde{z} = 0$. It follows that w lies in $U \cap V$ if and only if it is of the form

$$w = (x, 0, 0) = x(1, 0, 0),$$

whence

$$U \cap V = \{(x, 0, 0) : x \in \mathbb{R}\} = \operatorname{span}\{(1, 0, 0)\}.$$

b) The dimension of X is 3, so any basis for X must have exactly 3 vectors. Since $|B_1 \cup B_2| = 4$, $B_1 \cup B_2$ cannot be a basis for X. This can also be observed directly, as

$$(1, 0, 1) = (1, 0, 0) + (0, 0, 1)$$

shows linear dependence.

c) The dimension of a vector space is simply the size of a basis for that vector space. Thus, $\dim U = \dim V = 2$ and $\dim U \cap V = 1$, which is consistent with the given formula.

(More generally one has $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$, but in this case $U + V = X$.)

$\boxed{2}$ **a)** We only show this for U, as the proof for V is almost exactly the same.

It is sufficient to show that U is closed under the vector space operations, and that it contains zero. That it contains the zero polynomial is immediate, because $0(1) = 0(2) = 0$. Let now $p, q \in U$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha p + \beta q)(1) = \alpha p(1) + \beta q(1) = \alpha \cdot 0 + \beta \cdot 0 = 0$$
$$(\alpha p + \beta q)(2) = \alpha p(2) + \beta q(2) = 0$$

shows that $(\alpha p + \beta q) \in U$. Hence, U is a subspace of $\mathcal{P}_3$.

b) The polynomials in U are those that have roots in 1 and 2, while the polynomials in V are those with roots in 1 and 3. The polynomials that are both in U and V are therefore those that have roots in the three points 1, 2 and 3. Written out,

$$U \cap V = \{p \in \mathcal{P}_3 : p(1) = p(2) = p(3) = 0\}.$$

A polynomial has a root in a point a if and only if $(t - a)$ is a factor of the polynomial. It follows that the polynomials in $U \cap V$ are precisely those on the form

$$p(t) = \alpha(t-1)(t-2)(t-3)$$

for $\alpha \in \mathbb{R}$.

c) Write the Lagrange basis for $\mathcal{P}_3$ with nodes in the points $1, 2, 3, 4$ as L_1, L_2, L_3, L_4. A Lagrange basis yields the simplest description of these subspaces. (A different choice for the fourth node will yield different bases, but is of course equally valid.)

A polynomial $p \in \mathcal{P}_3$ can then be written

$$p(t) = a_1 L_1(t) + a_2 L_2(t) + a_3 L_3(t) + a_4 L_4(t).$$

Requiring that p lie in U is then equivalent to

$$p(1) = a_1 = p(2) = a_2 = 0,$$

and so a basis for U is $B_U = \{L_3, L_4\}$. Similarly, bases for V and $U \cap V$ are $B_V = \{L_2, L_4\}$ and $B_{U\cap V} = \{L_4\}$, respectively.

(Please note that it is *not* always the case that $B_{U\cap V} = B_U \cap B_V$!)

$\boxed{3}$ **a)** An infinite family is said to be linearly independent if every finite sub-family is linearly independent. Let therefore $t^{i_0}, t^{i_1}, \ldots, t^{i_m}$ be a finite sub-family, where we may assume that $i_0 < i_1 < \cdots < i_m$, and suppose that

$$a_{i_0} t^{i_0} + a_{i_1} t^{i_1} + \cdots + a_{i_m} t^{i_m} = 0 \tag{2}$$

for some real numbers $a_{i_0}, a_{i_1}, \ldots, a_{i_m}$ and all $t \in \mathbb{R}$. We wish to show that the coefficients a_{i_j} are all zero, which we will do by induction on j.

First, observe that division by t^{i_0} in (2) yields

$$a_{i_0} + a_{i_1} t^{i_1 - i_0} + \cdots + a_{i_m} t^{i_m - i_0} = 0$$

for $t \neq 0$. Taking the limit as $t \to 0$ shows that $a_{i_0} = 0$. Next, suppose that $a_{i_0} = a_{i_1} = \cdots = a_{i_{k-1}} = 0$. Then division by t^{i_k} in (2) implies that

$$a_{i_k} + a_{i_{k+1}} t^{i_{k+1} - i_k} + \cdots + a_{i_m} t^{i_m - t_k} = 0,$$

again for $t \neq 0$. Hence, also $a_{i_k} = 0$. Thus all the a_{i_j} are zero, which shows that the finite sub-family is linearly independent.

(If one views polynomials as abstract expressions, then the linear independence is immediate. Here, polynomials are viewed as real-valued functions on $\mathbb{R}$.)

b) For a general polynomial $p \in \mathcal{P}$, which we write as

$$p(t) = \sum_{i=0}^{n} a_i t^i,$$

the condition $p(t) = p(-t)$ becomes

$$p(t) = \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} (-1)^i a_i t^i = p(-t)$$

or

$$\sum_{i=0}^{n} (1 - (-1)^i) a_i t^i = 0.$$

Since $1 + (-1)^i$ is 0 when i is even and 2 when i is odd, this is equivalent to

$$\sum_{i=0}^{k} a_{2i+1} t^{2i+1} = 0,$$

where $2k + 1$ is the greatest odd number less than or equal to n. From **a)** we know that the family $\{1, t, t^2, \ldots\}$ is linearly independent, so this shows that the even polynomials are precisely the polynomials consisting of only even powers of t. This also shows that U_e is a subspace of $\mathcal{P}$, because

$$U_e = \operatorname{span}\{t^i : i = 0, 2, 4, \ldots\}.$$

For any function $f\colon \mathbb{R} \to \mathbb{R}$, we may write

$$f(t) = f_e(t) + f_o(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2},$$

which uniquely decomposes f into an even and an odd function. For polynomials, this is equivalent to splitting the even and odd powers of t. This means that each equivalence class (also called a coset in this case) $p + U_e$ has a unique odd member, namely p_o. This leads to identifying the quotient space $\mathcal{P}/U_e$ with the odd polynomials:

Let

$$U_o = \operatorname{span}\{t^i : i = 1, 3, 5, \ldots\}$$

be the subspace of odd polynomials, and define $T\colon U_o \to \mathcal{P}/U_e$ by $Tp = p + U_e$, which is clearly linear. If $Tp = p + U_e = 0 + U_e$, then $p \in U_e$. The only polynomial that is both even and odd is 0, so $p = 0$ and T is injective. It is onto because if $p + U_e = p_o + U_e$ is an arbitrary element of $\mathcal{P}/U_e$, then $Tp_o = p_o + U_e$. This shows that $\mathcal{P}/U_e$ is isomorphic to U_o.

Remark: It is a general theorem that if U, V are subspaces of a vector space W and $W = U \oplus V$, then the quotient space W/U is isomorphic to V. Here the notation $U \oplus V$ means the sum of two vector spaces such that $U \cap V = \{0\}$, also known as a *direct sum*. A vector space W is a direct sum of subspaces U and V if and only if every element of W can be written *uniquely* as the sum of an element of U and an element of V. The proof of this theorem is essentially what was done here, only for general vector spaces (this changes very little). The splitting of polynomials into odd and even parts shows that $\mathcal{P} = U_e \oplus U_o$.

c) For a general polynomial p, the condition $p(0) = 0$ is equivalent to a vanishing constant term. It follows that

$$V = \operatorname{span}\{t^i : i = 1, 2, \ldots\},$$

and since

$$\begin{aligned}\mathcal{P} &= \operatorname{span}\{t^i : i = 0, 1, 2, \ldots\} \\ &= \operatorname{span}\{t^i : i = 0\} \oplus \operatorname{span}\{t^i : i = 1, 2, \ldots\} \\ &= \operatorname{span}\{1\} \oplus V,\end{aligned}$$

we may identify $\mathcal{P}/V$ with the constant polynomials (see the remark in **b)**), which may further be identified with the real numbers. The quotient space $\mathcal{P}/V$ is therefore one-dimensional.

(An explicit isomorphism between $\mathbb{R}$ and $\mathcal{P}/V$ is $S\colon \mathbb{R} \to \mathcal{P}/V$ defined by $Sa_0 = a_0 + V$.)