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**Solutions to exercise set 3**

Note that there are several ways to approach the problems, but that only one is presented here. Other solutions may be just as valid!

**1** **a)** This linear system is much simpler to solve than a general one, due to all the structure in the coefficients. Observe that

$$\begin{aligned}
(1)+(3):&\quad 2a_1 + 2a_3 = f_1 + f_3 &(5)\\
(1)-(3):&\quad 2a_2 + 2a_4 = f_1 - f_3 &(6)\\
(2)+(4):&\quad 2a_1 - 2a_3 = f_2 + f_4 &(7)\\
(2)-(4):&\quad 2ia_2 - 2ia_4 = f_2 - f_4. &(8)
\end{aligned}$$

Repeating a similar procedure again, we find

$$\begin{aligned}
(5)+(7):&\quad 4a_1 = f_1 + f_2 + f_3 + f_4\\
(5)-(7):&\quad 4a_3 = f_1 - f_2 + f_3 - f_4\\
(6)-i(8):&\quad 4a_2 = f_1 - if_2 - f_3 + if_4\\
(6)+i(8):&\quad 4a_4 = f_1 + if_2 - f_3 - if_4,
\end{aligned}$$

where it only remains to divide by 4.

**b)** The matrix $B$ is given by

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i, \end{bmatrix}$$

and from **a)** we read that

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i, \end{bmatrix}$$

which is equal to $\frac{1}{4}\overline{B} = \frac{1}{4}B^*$.

**2** **a)** From the first question we know that $F_4 = B/2$ is unitary, which means that $F_4^* F_4 = I$, where $I$ is the identity matrix. But this shows that the columns of

$F_4$ are orthonormal in $\mathbb{C}^4$, because we also have

$$F_4^* F_4 = \begin{bmatrix} u_1^* \\ u_2^* \\ u_3^* \\ u_4^* \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_2, u_1 \rangle & \langle u_3, u_1 \rangle & \langle u_4, u_1 \rangle \\ \langle u_1, u_2 \rangle & \langle u_2, u_2 \rangle & \langle u_3, u_2 \rangle & \langle u_4, u_2 \rangle \\ \langle u_1, u_3 \rangle & \langle u_2, u_3 \rangle & \langle u_3, u_3 \rangle & \langle u_4, u_3 \rangle \\ \langle u_1, u_4 \rangle & \langle u_2, u_4 \rangle & \langle u_3, u_4 \rangle & \langle u_4, u_4 \rangle \end{bmatrix}.$$

**b)** In order to simplify this, we make the observation that

$$F_4 = F_4^T = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \\ u_4^T \end{bmatrix} = \begin{bmatrix} u_1^* \\ u_4^* \\ u_3^* \\ u_2^* \end{bmatrix}. \tag{9}$$

We can now compute

$$F_4^2 = \begin{bmatrix} u_1^* \\ u_4^* \\ u_3^* \\ u_2^* \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_4, u_1 \rangle & \langle u_3, u_1 \rangle & \langle u_2, u_1 \rangle \\ \langle u_1, u_2 \rangle & \langle u_4, u_2 \rangle & \langle u_3, u_2 \rangle & \langle u_2, u_2 \rangle \\ \langle u_1, u_3 \rangle & \langle u_4, u_3 \rangle & \langle u_3, u_3 \rangle & \langle u_2, u_3 \rangle \\ \langle u_1, u_4 \rangle & \langle u_4, u_4 \rangle & \langle u_3, u_4 \rangle & \langle u_2, u_4 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which in turn can be used to compute $F_4^3$. This matrix simply reorders the columns of another matrix if multiplied on the right. Namely,

$$\begin{aligned} F_4^3 = F_4 F_4^2 &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} u_1 & u_4 & u_3 & u_2 \end{bmatrix} \\ &= F_4^*, \end{aligned}$$

where the last equality comes from the observation in (9). It finally follows that

$$F_4^4 = F_4 F_4^3 = F_4 F_4^* = I,$$

since $F_4$ is unitary.

**3** Linear independence of the $v_i$ means, by definition, that if

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

for scalars $\alpha_i$, then all the $\alpha_i$ are zero. Stated in a different way, if the $\alpha_i$ are *not* all zero, then

$$\sum_{i=1}^{n} \alpha_i v_i \neq 0.$$

In particular, for any fixed $i$ we can choose $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$ to obtain

$$v_j \neq 0.$$

Hence all the $v_i$ are different from the zero vector.

4 Suppose that $m > n$ (we show the contrapositive). Since the $v_i$ span $X$, we can find scalars $\alpha_{ij}$ such that

$$u_i = \sum_{j=1}^{n} \alpha_{ij} v_j$$

for $i = 1, \ldots, m$. The matrix $A = (\alpha_{ij})$ has $m$ rows and $n$ columns, and since $m > n$ the rows of A (which lie in $\mathbb{K}^n$) must be linearly dependent. We can therefore find scalars $\beta_i$, not all zero, such that

$$\sum_{i=1}^{m} \beta_i \alpha_{ij} = 0$$

for $j = 1, \ldots, n$. But then

$$\begin{aligned} \sum_{i=1}^{m} \beta_i u_i &= \sum_{i=1}^{m} \beta_i \sum_{j=1}^{n} \alpha_{ij} v_j \\ &= \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \beta_i \alpha_{ij} \right) v_j \\ &= 0 \end{aligned}$$

shows that the $u_i$ are linearly dependent. (Hence if the $u_i$ are linearly independent, then $m \leq n$.)

5 Every vector space has a basis, so let $v_1, \ldots, v_n$ be a basis for $V$. Next, define the map $T \colon \mathbb{R}^n \to V$ by

$$Tx = \sum_{i=1}^{n} x_i v_i.$$

We need to verify that $T$ is linear, injective and surjective. These properties together will be imply that $T$ is an isomorphism (of vector spaces), and therefore that $V$ is isomorphic to $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, we find

$$\begin{aligned} T(\alpha x + \beta y) &= \sum_{i=1}^{n} (\alpha x_i + \beta y_i) v_i \\ &= \alpha \sum_{i=1}^{n} x_i v_i + \beta \sum_{i=1}^{n} y_i v_i \\ &= \alpha Tx + \beta Ty, \end{aligned}$$

which shows that $T$ is linear.

For surjectivity, let $u \in V$ be arbitrary. Since the $v_i$ form a basis, there are scalars $x_i$ such that

$$u = \sum_{i=1}^{n} x_i v_i.$$

The right hand side of this equation is simply $Tx$, so $Tx = u$ and $T$ is surjective.

Finally, if

$$0 = Tx = \sum_{i=1}^{n} x_i v_i,$$

then all the $x_i$ must be zero by linear independence of the $v_i$. Thus $x = 0$, and so $T$ is injective. A linear map being injective is equivalent to it having a trivial kernel (why?).