TMA4145 Linear Methods

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Exercise set 3

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Please justify all the steps in your answers!

1 Consider the following linear system:

$$
\begin{aligned}
a_1 + \ \ a_2 + a_3 + \ \ a_4 &= f_1 && (1)\\
a_1 + ia_2 - a_3 - ia_4 &= f_2 && (2)\\
a_1 - \ \ a_2 + a_3 - \ \ a_4 &= f_3 && (3)\\
a_1 - ia_2 - a_3 + ia_4 &= f_4 && (4)
\end{aligned}
$$

a) Solve the system.

b) If we write the system above as

$$Ba = f,$$

what is the relationship between B and B^{-1}?

2 Let F_4 be the Fourier matrix

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i. \end{bmatrix}$$

a) Show that the columns u_i, $i = 1, 2, 3, 4$ of F_4 are orthonormal ($\langle u_i, u_j \rangle = \delta_{ij}$) in the usual inner product

$$\langle x, y \rangle = \sum_{i=1}^{4} x_i \bar{y}_i$$

on $\mathbb{C}^4$.
(This should be shown without directly computing all the inner products.)

b) Compute the powers F_4^2, F_4^3 and F_4^4 of F_4.
(You are not meant to brute-force this.)

3 Suppose that the vectors $v_1, \dots, v_n$ are linearly independent in a vector space V. Show that the zero vector is not among the v_i.

4 Suppose that $v_1, \ldots, v_n$ span a vector space V, and suppose that $u_1, \ldots, u_m$ are linearly independent in V. Show that $m \leq n$.

You may assume that the result is true in $\mathbb{K}^k$ for $k \geq 1$, where $\mathbb{K}$ is the scalar field of V (which is typically proved before this result).

5 Let V be a finite-dimensional real vector space. Show that V is isomorphic to $\mathbb{R}^n$.