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TMA4145 Linear Methods Autumn 2014

Solutions to exercise set 9

1 Define T by

$$T(\sum_{k=1}^{n} c_k v_k) = (c_1 + c_2)v_1 + \cdots + (c_{n-1} + c_n)v_{n-1} + c_n v_n.$$

It is a linear transformation. More generally, for any $u_j \in V$, there is a unique linear transformation T such that $T(v_j) = u_j$.

The columns of the matrix of T in the basis $\{v_1, \ldots, v_n\}$ are coefficients of $T(v_1), \ldots, T(v_n)$ in this basis,

$$\begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

2 The range $\mathrm{ran}(T)$ is $P(\mathbb{C})$. We know that the range of T is a subspace of $P(\mathbb{C})$; if it contains all monomials t^j, then it contains all their linear combinations. Clearly, $T(t^{j+k}) = (j+k)!(j!)^{-1}t^j$. Thus by linearity $t^j \in \mathrm{ran}(T)$.

It is easy to prove by induction (or by looking at polynomial coefficients) that $p^{(k)} = 0$ if and only $\deg p \leq k-1$. Thus $\ker(T) = P_{k-1}(\mathbb{C})$, the space of polynomials of degree less than or equal to $k-1$.

3 The rank–nullity theorem says that $\dim(\ker(T)) + \dim(\mathrm{ran}(T)) = n$. Let $\{e_1, \ldots, e_k\}$ be a basis for $\ker(T)$ and $\{f_1, \ldots, f_j\}$ be a basis for $\mathrm{ran}(T)$.

$1 \Rightarrow 2$: If $\mathrm{ran}(T) \cap \ker(T) = \{0\}$ then $\{e_1, \ldots, e_k, f_1, \ldots, f_j\}$ is a linearly independent set of elements in V. It is a basis since the number of elements is equal to the dimension of V. Thus $\mathrm{ran}(T) + \ker(T) = V$.

$2 \Rightarrow 1$: If $\mathrm{ran}(T) + \ker(T) = V$ then $\{e_1, \ldots, e_k, f_1, \ldots, f_j\}$ span V, but the number of elements in this set is equal to the dimension of V, thus they are linearly independent.

We have proved that $1 \Leftrightarrow 2$, and 1 and 2 simultaneously imply 3.

4 **a)** If $y \in \mathrm{ran}(\mathrm{T}^2)$, then there exists $x \in V$ such that $y = T^2(x) = T(T(x))$ then $y \in \mathrm{ran}(\mathrm{T})$. Thus $\mathrm{ran}(\mathrm{T}^2) \subset \mathrm{ran}(\mathrm{T})$. This implies

$$\mathrm{rank}(T^2) = \dim(\mathrm{ran}(T^2)) \leq \dim(\mathrm{ran}(T)) = \mathrm{rank}(T).$$

b) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, then $\operatorname{ran}(\mathrm{T}^2) = \operatorname{ran}(\mathrm{T})$, $T : \operatorname{ran}(T) \to \operatorname{ran}(T^2)$ is onto and since the spaces coincide, it is one-to-one. Thus $T(y) \neq 0$ when $y \in \operatorname{ran}(T) \setminus \{0\}$. We get $\operatorname{ran}(T) \cap \ker(T) = \{0\}$.

c) Consider the chain of subspaces $\operatorname{ran}(T) \supset \operatorname{ran}(T^2) \supset \operatorname{ran}(T^3) \supset \cdots$. All of them have finite dimension and from some element, the dimension will be constant, i.e., the exists $k \geq 1$ such that $\operatorname{ran}(T^k) = \operatorname{ran}(T^m)$ for any $m \geq k$. In particular $\operatorname{ran}(T^k) = \operatorname{ran}(T^{2k})$, applying **b)** to $T_1 = T^k$, we obtain $\operatorname{ran}(T_1) \cap \ker(T_1) = \{0\}$ and then by Problem 3, $V = \operatorname{ran}(T^k) \oplus \ker(T^k)$.

$\boxed{5}$ Suppose that $aS + bT = 0$ and $b \neq 0$, then $aSx + bTx = 0$ for any $x \in X$. Since T is a non-zero transformation, there exists x such that $Tx \neq 0$. We have $Tx = ab^{-1}Sx \in \operatorname{ran}(S)$ since $\operatorname{ran}(S)$ is a subspace. Thus $Tx \in \operatorname{ran}S \cap \operatorname{ran}(T)$ and $Tx = 0$, which leads to a contradiction. Hence if $aS + bT = 0$ then $b = 0$, similarly $a = 0$ and S and T are linearly independent.

$\boxed{6}$ **a)** A map $g\colon M \to M$ on a metric space is a contraction if there is some $\alpha < 1$ such that $d(g(x), g(y)) \leq \alpha d(x, y)$ for all $x, y \in M$. In this case, we have $g(x) = Tx + b$, and $d(u, v) = \|u - v\|$. Substituting in, we obtain the condition that there is some $\alpha < 1$ such that

$$\|Tx + b - (Ty + b)\| \leq \alpha \|x - y\|$$

which simplifies to $\|T(x - y)\| \leq \alpha\|x - y\|$. We can simplify this further to $\|Tz\| \leq \alpha\|z\|$ by noting that if the former holds, then it holds for $y = 0$, and if the latter holds then it holds for $z = x - y$. Thus the condition is that there is some $\alpha < 1$ such that $\|Tz\| \leq \alpha\|z\|$ for all $z \in X$.

b) The map $x \mapsto Ix + b$ is not a contraction since $I(e_1) = e_2$ and thus $\|Ie_1\| = 1 = \|e_1\|$. However, $x \mapsto I(Ix + b) + b$ is a contraction. This is of the form $x \mapsto Tx + c$, where $T = I^2$ (and $c = Ib + b$).
Written out, I^2 is

$$I^2(\xi_n) = \left(0, 0, \frac{1}{2}\xi_1, \frac{1}{6}\xi_2, \frac{1}{12}\xi_3, \dots\right)$$

from which we see that each of the coefficients is multiplied by $\frac{1}{k(k-1)}$, and thus $\left\|I^2(x)\right\| \leq \frac{1}{2}\|x\|$, with equality when $x = e_1$. Hence $x \mapsto I(Ix + b) + b$ is a contraction with contraction constant $\frac{1}{2}$.

c) The first five iterations with $x_0 = 0$ are:

$$\begin{aligned}
&(1, 0, 0, 0, 0, 0, \dots)\\
&(1, 1, 0, 0, 0, 0, \dots)\\
&(1, 1, \frac{1}{2}, 0, 0, 0, \dots)\\
&(1, 1, \frac{1}{2}, \frac{1}{6}, 0, 0, \dots)\\
&(1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, 0, \dots).
\end{aligned}$$

The first five iterations with $x_0 = (\xi_n)$ are:

$$\begin{aligned}
&(1, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \frac{1}{4}\xi_4, \frac{1}{5}\xi_5, \dots)\\
&(1, 1, \frac{1}{2}\xi_1, \frac{1}{6}\xi_2, \frac{1}{12}\xi_3, \frac{1}{20}\xi_4, \dots)\\
&(1, 1, \frac{1}{2}, \frac{1}{6}\xi_1, \frac{1}{24}\xi_2, \frac{1}{60}\xi_3, \dots)\\
&(1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}\xi_1, \frac{1}{120}\xi_2, \dots)\\
&(1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}\xi_1, \dots).
\end{aligned}$$

The iterations seem to get close very quickly, the dependence on the initial vector is very small.

$\boxed{7}$ **a)** Augmenting the matrix A with the identity matrix, we get

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 0 \\ 6 & 9 & 8 & 0 & 0 & 1 \end{bmatrix}.$$

Adding $(-3)\times$ row 1 to row 3 gives

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{bmatrix},$$

meaning that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

This can be expressed as $\tilde{L}A = U$. Now invert the matrix $\tilde{L}$:

$$L := \tilde{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Then

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

Both of these matrices are invertible (see **b)**), $\ker A = \{0\}$ and $\operatorname{ran} A = \mathbb{R}^3$ (rank–nullity theorem, Fredholm alternative).

b) We want to solve

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

in x for arbitrary y; this amounts to finding A^{-1}. The inverse of U is

$$U^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{10} & -\frac{3}{5} \\ 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 0 & -1 \end{bmatrix},$$

and that of L is $\tilde{L}$. Hence,

$$A^{-1} = U^{-1}\tilde{L} = \begin{bmatrix} \frac{23}{10} & -\frac{3}{10} & -\frac{3}{5} \\ -\frac{21}{5} & \frac{1}{5} & \frac{7}{5} \\ 3 & 0 & -1 \end{bmatrix},$$

and the answer is

$$\begin{aligned} x_1 &= \frac{23}{10}y_1 - \frac{3}{10}y_2 - \frac{3}{5}y_3 \\ x_2 &= -\frac{21}{5}y_1 + \frac{1}{5}y_2 + \frac{7}{5}y_3 \\ x_3 &= 3y_1 - y_3. \end{aligned}$$

c) Writing the vectors in row form, let

$$e_1 := \frac{A_1}{\|A_1\|} = \frac{1}{\sqrt{40}}(2,0,6) = \frac{1}{\sqrt{10}}(1,0,3),$$

and define

$$v_2 := A_2 - \langle A_2, e_1\rangle e_1 = (3,5,9) - \frac{\langle (3,5,9),(1,0,3)\rangle}{10}(1,0,3) = (0,5,0).$$

Let $e_2 := (0,1,0)$. Finally,

$$\begin{aligned} v_3 &:= A_3 - \langle A_3, e_1\rangle e_1 - \langle A_3, e_2\rangle e_2 \\ &= (3,7,8) - \frac{\langle (3,7,8),(1,0,3)\rangle}{10}(1,0,3) - \langle (3,7,8),(0,1,0)\rangle (0,1,0) \\ &= \frac{1}{10}\left((30,70,80) - (27,0,81) - (0,70,0)\right) = \frac{1}{10}(3,0,-1) \end{aligned}$$

yields the last orthonormal basis vector $e_3 := \frac{1}{\sqrt{10}}(3,0,-1)$. According to the Gram–Schmidt orthonormalisation algorithm,

$$\{e_1, e_2, e_3\} = \left\{\frac{1}{\sqrt{10}}(1,0,3), (0,1,0), \frac{1}{\sqrt{10}}(3,0,-1)\right\}$$

is an orthonormal basis for $\mathbb{R}^3$ obtained from the (linearly independent) vectors A_1, A_2, A_3.