Norwegian University of Science and Technology
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TMA4145 Linear Methods Autumn 2014

Exercise set 9

Read 4.3–4.4 of Chapter 4 in Lecture Notes.

1 **Friedberg et al, 2.2 problem 10**

Let $\{v_1, \dots, v_n\}$ be a basis for V. Show that there exists a linear transformation $T : V \to V$ such that $T(v_1) = v_1$, $T(v_j) = v_{j-1} + v_j$, $j = 2, \dots, n$. Find the matrix of this transformation in the basis $\{v_1, \dots, v_n\}$.

2 Let $P(\mathbb{C})$ be the vector space of all polynomials and let $T(p) = p^{(k)}$ be the k-th derivative of p. Find the range and the kernel of T.

3 Let V be a finite dimensional vector space and T be a linear transformation. Show that the following three statements are equivalent

1. $\operatorname{ran}(T) \cap \ker(T) = \{0\}$,
2. $\operatorname{ran}(T) + \ker(T) = V$, i.e., for any $x \in V$ there exists $y_1 \in \operatorname{ran}(T)$ and $y_2 \in \ker(T)$ such that $x = y_1 + y_2$,
3. $\operatorname{ran}(T) \oplus \ker(T) = V$.

Be sure you use that V is finite dimensional.

Hint: Apply the rank-nullity theorem.

4 **Friedberg et al, 2.3 problem 16**

Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove the following statements.

a) $\operatorname{ran}(\mathrm{T}^2) \subset \operatorname{ran}(\mathrm{T})$ and $\operatorname{rank}(T) \geq \operatorname{rank}(T^2)$.

b) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, then $\operatorname{ran}(T) \cap \ker(T) = \{0\}$.
Hint: Consider $T : \operatorname{ran}(T) \to \operatorname{ran}(T^2)$.

c) $V = \operatorname{ran}(T^k) \oplus \ker(T^k) = \{0\}$ for some positive integer k.
Hint: Use **b)** and Problem 2.

5 Problem 5, Friedberg et al, 2.2 problem 13

Let X and Y be vector spaces and T and S be linear transformations from X to Y. Suppose that $\operatorname{ran}(T) \cap \operatorname{ran}(S) = \{0\}$. Show that $\{S, T\}$ is a linear independent set in $L(X, Y)$.

6 Problem 2 from continuation Exam 2009

a) Let $(X, \|\cdot\|)$ be a Banach space. Let $T\colon X \to X$ be a continuous linear transformation. Let $b \in X$ be a fixed vector.

Consider the map $X \to X$ given by $x \mapsto Tx + b$. Express as simply as you can the condition that is needed for this to be a contraction.

b) Recall that l_∞ is the space of all *bounded* sequences of real numbers with the supremum norm,

$$\|(\xi_n)\|_\infty = \sup\{|\xi_n| : n \in \mathbb{N}\}.$$

Let $I\colon l_\infty \to l_\infty$ be the map

$$I(\xi_n) = \left(\frac{1}{n}\xi_{n+1}\right).$$

For example,

$$I(1, 1, \dots) = \left(0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right).$$

Let $b \in l_\infty$ be the sequence $(1, 0, 0, \dots)$. Prove that the map $x \mapsto Ix + b$ is not a contraction on l_∞, but that $x \mapsto I(Ix + b) + b$ is a contraction. What is the contraction constant?

c) Carry out the first five iterations (of $x \mapsto Ix + b$) starting from $x_0 = 0$. Then carry out the first five iterations starting from an arbitrary sequence $x_0 = (\xi_1, \xi_2, \dots)$. What do you notice?

7 Problem 2 from Exam 2012

a) Find an LU-decomposition of

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}.$$

What is $\ker A$ and $\operatorname{ran} A$?

b) Determine the coordinates $(x_1, x_2, x_3)_A$ of an arbitrary vector $y = (y_1, y_2, y_3)$ in the basis $\{A_1, A_2, A_3\}$ given by the column vector of A.

c) Perform a Gram–Schmidt orthonormalisation of $\{A_1, A_2, A_3\}$.