Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods Autumn 2014

Exercise set 6

Read Sections 3.4-6 of Chapter 3 in Lecture Notes.

1 **Kreyszig 3.1, problem 7**

Suppose that X is an inner-product space, $u, v \in X$ are such that $\langle u, x\rangle = \langle v, x\rangle$ for any $x \in X$. Show that $u = v$.

2 Let vectors $v_1, \dots, v_k$ in an inner-product space X be pair-wise orthogonal. Show that they are linearly independent.

Hint: Assume that $a_1 v_1 + \cdots + a_k v_k = 0$ and use the orthogonality to prove that $a_1 = \cdots = a_k = 0$.

3 Show that l_1 with the usual norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$ is not an inner-product space, i.e., there is no inner product in l_1 compatible with the norm $\|\cdot\|_1$.

Hint: use the polarization identity.

4 **Kreyszig 3.1, problem 15**

Let X be a finite dimensional vector space with a basis $\{e_j\}_{j=1}^n$. Show that an inner product on X is uniquely determined by its values on the pairs of basis vectors, $c_{ij} = \langle e_i, e_j\rangle$. Can the values c_{ij} be chosen arbitrarily?

5 **a)** Suppose that X is a normed space, show that the unit ball $B_1(0) = \{x \in X : \|x\| < 1\}$ is a convex set.

(Recall that a subset S of a vector space X is called *convex* if for any $x, y \in S$ and any $t \in (0, 1)$, $tx + (1-t)y \in S$.)

b) Let S be a convex bounded non-empty open centrally symmetric (meaning that $x \in S \Rightarrow -x \in S$) set in $\mathbb{R}^2$. Define $\|x\|_S = \inf\{c > 0 : c^{-1}x \in S\}$. Prove that it is a norm in $\mathbb{R}^2$ and the unit ball in this norm is S.

6 Problem 3 from cont. exam 2009

Let $(X, \|\cdot\|)$ be a normed vector space.

a) Let (a_n) be a sequence of points in X with the property that the series $\sum \|a_n\|$ converges in $\mathbb{R}$. Prove that the sequence (s_n) in X defined by

$$s_n = \sum_{k=1}^{n} a_k$$

is Cauchy.

b) Prove that a normed vector space $(X, \|\cdot\|)$ is complete if and only if whenever (a_n) is a sequence in X with the property that $\sum \|a_n\|$ converges, then the sequence of partial sums (s_n) with $s_n = \sum_{k=1}^{n} a_k$ converges in X.
You may assume that a Cauchy sequence (s_n) in X has a subsequence (s_{n_m}) with the property that $\|s_{n_{m+1}} - s_{n_m}\| < \frac{1}{2^m}$.

7 Problem 5 from Exam 2006

Consider the subspace

$$M = \left\{ x | x(t) = 0 \text{ for } 0 \leqslant t \leqslant \frac{1}{2} \right\}$$

of $C[0,1]$, and let $C[0,1]$ have the norm derived from the inner product given by

$$\langle x, y \rangle = \int_0^1 x(t)\overline{y(t)}\,\mathrm{d}t.$$

a) Show that if $x \in C[0,1]$ and $y \in M$, then

$$\int_0^{\frac{1}{2}} |x(t)|^2\,\mathrm{d}t \leqslant \|x - y\|^2.$$

Show that M is a closed subset of $C[0,1]$.

b) Show that $\|x - 1\| \geqslant \frac{1}{\sqrt{2}}$ for all $x \in M$. Does there exist an element $x_0 \in M$ such that $\|x_0 - 1\| = \frac{1}{\sqrt{2}}$?
(Here 1 denotes the constant function $1(t) = 1$ for $0 \leqslant t \leqslant 1$.)