Norwegian University of Science and Technology
Department of Mathematical Sciences

Suggested Solutions
TMA4145, December 2012

Problem 1 *(Overview)*
For each of the following, indicate whether it is true or false (no proof required).

(i) Any vector space contains a zero element.

(ii) $P_n(\mathbb{R}) \cong \mathbb{R}^n$.

(iii) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable.

(iv) In a normed space, all Cauchy sequences are convergent.

(v) $\{e^{ikx}\}_{k\in\mathbb{Z}}$ defines an orthonormal basis for $L_2((-\pi,\pi),\mathbb{C})$, $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

(vi) $\mathbb{C}^n$ is the dual of $\mathbb{R}^n$.

(vii) A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\ker(T) = \{0\}$.

(viii) The function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, is locally Lipschitz continuous.

(ix) Pythagoras theorem, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \perp y$, holds in inner-product spaces.

(x) The differential operator $1 + (\frac{d}{dx})^2$ is invertible on $C^\infty_{2\pi\text{-per}}(\mathbb{R})$.

Solution

(i) Yes, any vector space contains a zero element.

(ii) No, $P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$ ($\sum_{k=0}^{n} a_k x^k \mapsto (a_0, \ldots, a_n)$ is an isomorphism).

(iii) Yes, according to the spectral theorem, any symmetric matrix is diagonalizable.

(iv) No, only in complete normed spaces (Banach spaces) do all Cauchy sequences converge.

(v) Yes, as can be checked by direct computation, $\{e^{ikx}\}_{k\in\mathbb{Z}}$ defines an orthonormal basis for $L_2((-\pi,\pi),\mathbb{C})$, $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

(vi) No, the correct dual of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself (any functional on $\mathbb{R}^n$ is given by a dot product).

(vii) Yes, a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\ker(T) = \{0\}$.

(viii) Yes, the function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, is locally Lipschitz continuous, since $|x^2 - y^2| = |x+y||x-y| \le L|x-y|$ for x, y in any bounded set.

(ix) Yes, as can be checked by direct computation, Pythagora's theorem, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \perp y$, holds in inner-product spaces.

(x) No, the differential operator $1 + (\frac{d}{dx})^2$ is not invertible on $C^\infty_{2\pi\text{-per}}(\mathbb{R})$, since, in particular, it has a non-trivial kernel, $\{\sin, \cos\}$.

Problem 2 *(Finite-dimensional linear transformations)*

a) Find an LU-decomposition of

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}.$$

What is $\ker(A)$, $\operatorname{ran}(A)$?

b) Determine the coordinates $(x_1, x_2, x_3)_A$ of an arbitrary vector $y = (y_1, y_2, y_3)$ in the basis $\{A_1, A_2, A_3\}$ given by the column vectors of A.

c) Perform a Gram–Schmidt orthonormalization of $\{A_1, A_2, A_3\}$.

Solution

(a) Augmenting the matrix A with the identity matrix, we get

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 0 \\ 6 & 9 & 8 & 0 & 0 & 1 \end{bmatrix}.$$

Adding $(-3) \times$ row 1 to row 3 yields

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{bmatrix},$$

meaning that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

This can be expressed as $\tilde{L}A = U$. Now, invert the matrix $\tilde{L}$:

$$L := \tilde{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Then

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since both these matrices are invertible (see (b)), $\ker(A) = \{0\}$ and $\operatorname{ran}(A) = \mathbb{R}^3$ (rank–nullity theorem, Fredholm alternative).

(b) We want to solve

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

in x for arbitrary y; this amounts to finding A^{-1}. The inverse of U is

$$U^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{10} & -\frac{3}{5} \\ 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 0 & -1 \end{bmatrix},$$

and that of L is $\tilde{L}$. Hence,

$$A^{-1} = U^{-1}\tilde{L} = \begin{bmatrix} \frac{23}{10} & -\frac{3}{10} & -\frac{3}{5} \\ -\frac{21}{5} & \frac{1}{5} & \frac{7}{5} \\ 3 & 0 & -1 \end{bmatrix},$$

and the answer is

$$\begin{aligned} x_1 &= \frac{23}{10}y_1 - \frac{3}{10}y_2 - \frac{3}{5}y_3, \\ x_2 &= -\frac{21}{5}y_1 + \frac{1}{5}y_2 + \frac{7}{5}y_3, \\ x_3 &= 3y_1 - y_3. \end{aligned}$$

(c) Writing vectors in row form, let

$$e_1 := \frac{A_1}{|A_1|} = \frac{1}{\sqrt{40}}(2,0,6) = \frac{1}{\sqrt{10}}(1,0,3),$$

and define

$$v_2 := A_2 - \langle A_2, e_1\rangle e_1 = (3,5,9) - \frac{\langle (3,5,9),(1,0,3)}{10}\rangle(1,0,3) = (0,5,0).$$

Let $e_2 := (0,1,0)$. Finally,

$$\begin{aligned} v_3 &:= A_3 - \langle A_3, e_1\rangle e_1 - \langle A_3, e_2\rangle e_2 \\ &= (3,7,8) - \frac{\langle (3,7,8),(1,0,3)\rangle}{10}(1,0,3) - \langle\langle (3,7,8),(0,1,0)\rangle(0,1,0) \\ &= \frac{1}{10}\left((30,70,80) - (27,0,81) - (0,70,0)\right) = \frac{1}{10}(3,0,-1) \end{aligned}$$

yields the last orthonormal basis vector, $e_3 := \frac{1}{\sqrt{10}}(3,0,-1)$. According to the Gram–Schmidt-orthonormalization algorithm,

$$\{e_1, e_2, e_3\} = \left\{\frac{1}{\sqrt{10}}(1,0,3), (0,1,0), \frac{1}{\sqrt{10}}(3,0,-1)\right\}$$

is an orthonormal basis for $\mathbb{R}^3$ obtained from the (linearly independent) vectors A_1, A_2, A_3.

Problem 3 *(Spectral theory, differential equations)*

a) Express $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ in Jordan normal form TJT^{-1}.

b) Solve the initial-value problem $\dot{x} = Ax$, $x(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution.

(a) The characteristic polynomial of A is

$$p_A(\lambda) = (1-\lambda)(3-\lambda) + 1 = (\lambda - 2)^2,$$

whose only root is $\lambda = 2$, with algebraic multiplicity 2. The eigenvectors of A are given by

$$(A - 2I)v = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \quad v = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R},$$ [1]

forming a one-dimensional space; hence, we need generalized eigenvectors. Since

$$(A - 2I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we can pick any vector $v_2 \in \mathbb{R}^2 \setminus \operatorname{span}\{(1,1)\}$, say

$$v_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

to start a Jordan chain. We then construct the vector

$$v_1 := (A - 2I)v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as the second element in the Jordan chain starting with v_2; the corresponding change-of-basis matrix is

$$T := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

with corresponding Jordan block matrix

$$J := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

[1] In this problem, it does not matter if one considers $\mathbb{R}^n$ or $\mathbb{C}^n$ as the ambient space.

Finally, one may check that, indeed,

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

(b) Expressed in the Jordan basis, the fundamental matrix is given by

$$\begin{aligned} U := \exp\left(t \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right) &= \exp\left(t\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right) \\ &= \exp(2tI) \exp\left(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = e^{2t}\left(I + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Returning to the original basis, we find the fundamental solution

$$X := TUT^{-1} = e^{2t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix},$$

with $X|_{t=0} = I$. The solution of the specified initial-value problem is thus given by

$$x(t) = e^{2t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1+3t \\ 4+3t \end{bmatrix},$$

which can be verified directly from the equation.

Problem 4 *(Normed spaces, minimal distances)*
Let $M \subset X$ be a subset of normed space $(X, \|\cdot\|)$, $x_0 \in X$. In each of the cases listed, calculate $\inf_{y\in M} \|x_0 - y\|$ and determine whether there exists a unique $y_0 \in M$ such that $\|x_0 - y_0\| = \inf_{y\in M} \|x_0 - y\|$. If so, find y_0.

a) $X = \mathbb{R}^3$ with norm $\|x\| = (\sum_{j=1}^3 x_j^2)^{1/2}$, $M = \operatorname{span}\{(1,1,0)\}$, $x_0 = (1,2,3)$.

b) $X = l_\infty(\mathbb{R})$ with norm $\|x\|_{l_\infty} = \sup_{j\in\mathbb{N}} |x_j|$, $M = \{\{x_j\}_{j\in\mathbb{N}} \colon \lim_{j\to\infty} x_j = 0\}$, $x_0 = \{\frac{j}{j+1}\}_{j\in\mathbb{N}}$.

c) $X = L_2((0,1),\mathbb{R})$ with norm $\|x\| = (\int_0^1 |x(t)|^2\, dt)^{1/2}$, $M = \operatorname{span}\{1, t\}$, the function x_0 given by $x_0(t) = t^2$.

Solution

(a) Let

$$e_1 := \tfrac{1}{\sqrt{2}}(1,1,0),$$

so that $\{e_1\}$ is an orthonormal basis for M. Since—in the Hilbert space $\mathbb{R}^3$ with the inner product $\langle x, y\rangle = \sum_{j=1}^{3} x_j y_j$ (which corresponds to the given norm)—Fourier coefficients are best possible, we get

$$y_0 = \langle x_0, e_1\rangle e_1 = \tfrac{1}{2}\langle (1,2,3),(1,1,0)\rangle (1,1,0) = (\tfrac{3}{2},\tfrac{3}{2},0),$$

with

$$d^2 = \|(1,2,3)\|^2 - \left|\frac{1}{\sqrt{2}}\langle (1,2,3),(1,1,0)\rangle\right|^2 = \frac{19}{2},$$

so that the minimal distance $\inf_{y\in M}\|x_0 - y\| = \sqrt{\frac{19}{2}}$ is realized by $y_0 = (\frac{3}{2},\frac{3}{2},0) \in M$. (One can check the answer, for example, by minimizing $f(t) := \|(1-t, 2-t, 3)\|^2$.)

(b) For all $y \in M$, we have

$$\|x_0 - y\|_{l_\infty} = \sup_{j\in\mathbb{N}}\left|\frac{j}{j+1} - y_j\right| \geq \lim_{j\to\infty}\left|\frac{j}{j+1} - y_j\right| = |1-0| = 1.$$

The two elements $(0,0,0,\ldots)$ and $(\frac{1}{2},\frac{1}{3},\frac{1}{4}\ldots)$ in M both realize this distance;

$$\|x_0 - (0,0,0,\ldots)\|_{l_\infty} = \sup_{j\in\mathbb{N}}\left|\frac{j}{j+1}\right| = 1,$$

and

$$\|x_0 - (\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{4},\ldots)\|_{l_\infty} = \sup_{j\in\mathbb{N}}\left|\frac{j-1}{j+1}\right| = 1,$$

so the minimal distance $d = 1$ is attained, but there is no unique minimizing element $y_0 \in M$.

(c) $L_2((0,1),\mathbb{R})$ is a Hilbert space, and M is a finite-dimensional subspace, whence closed and convex (and non-empty); the existence of a unique minimizing element y_0 is thus guaranteed by the minimal distance theorem. We find y_0 by constructing an orthonormal basis $\{e_1, e_2\}$ for M. Since

$$\int_0^1 |1|^2\, dt = 1 \quad \text{let} \quad e_1 := 1.^2$$

[2]We adopt here, as in the problem formulation, the slight abuse of notation $f = f(t)$.

Then let

$$v_2 = t - \langle t, 1\rangle 1 = t - \int_0^1 t\,dt = t - \frac{1}{2},$$

and

$$e_2 = \frac{v_2}{\|v_2\|} = \sqrt{3}(2t-1)$$

Thus,

$$y_0 = \langle t^2, e_1\rangle e_1 + \langle t^2, e_2\rangle e_2 = \int_0^1 s^2\,ds + 3\left(\int_0^1 s^2(2s-1)\,ds\right)(2t-1),$$

and $y_0 = t - \frac{1}{6}$ is the unique minimizing element. The distance is given by

$$\begin{aligned} d^2 &= \|x_0\|^2 - |\langle t^2, e_1\rangle|^2 - |\langle t^2, e_2\rangle|^2 \\ &= \int_0^1 t^4\,dt - \left(\int_0^1 t^2\,dt\right)^2 - 3\left(\int_0^1 t^2(2t-1)\,dt\right)^2 \\ &= \frac{1}{5} - \frac{1}{9} - \frac{1}{12} = \frac{1}{180}, \end{aligned}$$

where the values of the integrals $\langle t^2, e_1\rangle$ and $\langle t^2, e_2\rangle$ are already known. Hence, $d = \frac{1}{6\sqrt{5}}$.

Problem 5 *(Hilbert spaces)*
Let $U \in B(H)$ be a unitary operator on a complex Hilbert space H, i.e., a bounded linear operator $H \to H$ such that

$$UU^* = U^*U = \mathrm{Id}.$$

Show that:

a) All eigenvalues of U satisfy $|\lambda| = 1$.

b) If $(U - \lambda)x = 0$, then $(U^* - \overline{\lambda})x = 0$.

c) Eigenspaces corresponding to different eigenvalues of U are orthogonal.

Solution

(a) Assume that $Ux = \lambda x$ for some $x \neq 0$. Then

$$|\lambda|^2\|x\|^2 = \lambda\overline{\lambda}\langle x, x\rangle = \langle \lambda x, \lambda x\rangle = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, x\rangle = \|x\|^2,$$

where we have used the properties of an inner product, the adjoint, and unitary operators. Since $x \neq 0$, we may divide by $\|x\|^2$, yielding that $|\lambda| = 1$ for all eigenvalues of a unitary operator U.

(b) The statement is trivial for $x = 0$, so assume that $x \neq 0$ is an eigenvector corresponding to the eigenvalue λ. Since $(U - \lambda)x = 0$, also

$$U^*(U - \lambda)x = (U^*U - \lambda U^*)x = (I - \lambda U^*)x = 0.$$

Multiplying with $\overline{\lambda}$, we get that

$$(\overline{\lambda} - \overline{\lambda}\lambda U^*)x = (\overline{\lambda} - U^*)x = 0,$$

since $|\lambda|^2 = 1$ according to (a). This proves the assertion.

(c) If $\lambda \neq \mu$ are two different eigenvalues with eigenvectors x and y, respectively, we have from (b) that

$$(\lambda - \mu)\langle x, y\rangle = \langle \lambda x, y\rangle - \langle x, \overline{\mu} y\rangle = \langle Ux, y\rangle - \langle x, U^*y\rangle = 0,$$

using the definition of U^*. Since $\lambda \neq \mu$, we must have $\langle x, y\rangle = 0$, so that $x \perp y$. Thus eigenspaces corresponding to different eigenvalues of U are orthogonal.

Problem 6 *(Banach spaces, isomorphisms)*
Let $P(\mathbb{R})$ denote the space of all polynomials $p(x) = \sum_k a_k x^k$ (finite sum) with real coefficients $a_0, a_1, \ldots$, equipped with the norm $\|p(x)\| = \max_k |a_k|$.

a) Find a Cauchy sequence in $P(\mathbb{R})$ that does not converge in $P(\mathbb{R})$.

b) Determine the completion of $P(\mathbb{R})$, i.e., find a normed space $\tilde{X}$ and a Banach space X such that $P(\mathbb{R}) \cong \tilde{X} \overset{\text{dense}}{\subset} X$. Here, '$\cong$' indicates an isometric isomorphism, i.e., an isomorphism between normed spaces.

Solution. This problem is greatly simplified, and essentially solved, by noting that the map

$$\phi\colon P(\mathbb{R}) \to l_0(\mathbb{R}), \qquad \sum_{k=0}^{\deg(p(x))} a_k x^k \mapsto (a_0, \ldots, a_{\deg(p(x))}, 0, 0, \ldots),$$

is an isomorphism from $P(\mathbb{R})$ to $l_0(\mathbb{R})$ (cf. Problem set 9), where

$$l_0(\mathbb{R}) = \{\{a_k\}_{k\geq 0} \in l_\infty(\mathbb{R})\colon a_k \neq 0 \text{ for finitely many } k\}.$$

Since, in addition,

$$\|p(x)\| = \max_{0\leq k\leq \deg(p(x))} |a_k| = \|\phi(p(x))\|_{l_\infty},$$

we have that ϕ is an isometric isomorphism: $(P(\mathbb{R}), \|\cdot\|) \cong (l_0(\mathbb{R}), \|\cdot\|_{l_\infty})$ as normed spaces.

(a) Define

$$x_n := (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, 0, \ldots), \qquad n \in \mathbb{N}.$$

Without loss of generality, let $m > n$. Then

$$\|x_m - x_n\|_{l_\infty} = \max_{n+1 \le j \le m} \{\tfrac{1}{j}\} = \tfrac{1}{n+1} \to 0 \quad \text{as} \quad m > n \to \infty,$$

so that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(l_0(\mathbb{R}), \|\cdot\|_{l_\infty})$. But there is no element in $P(\mathbb{R}) \cong l_0(\mathbb{R})$ that is the limit of $\{x_n\}_{n\in\mathbb{N}}$. Say there were one, $y = \{y_j\}_{j\in\mathbb{N}}$. Then, for any $j \le n$,

$$|\tfrac{1}{j} - y_j| \le \|x_n - y\|_{l_\infty} \to 0 \quad \text{as} \quad n \to \infty,$$

so that $y_j = \frac{1}{j}$ for all $j \in \mathbb{N}$. But this contradicts the assumption that $y \in l_0(\mathbb{R})$, in turn proving that $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $(l_0(\mathbb{R}), \|\cdot\|_{l_\infty})$.

(b) As proved in Problem set 8, the closure of $l_0(\mathbb{R})$ in $l_\infty(\mathbb{R})$ is $c_0(\mathbb{R})$, the space of sequences with a vanishing limit:

$$c_0(\mathbb{R}) = \{\{y_j\}_{j\in\mathbb{N}} \in l_\infty(\mathbb{R}) \colon \lim_{j\to\infty} y_j = 0\}.$$

Hence,

$$P(\mathbb{R}) \cong l_0(\mathbb{R}) \overset{\text{dense}}{\subset} c_0(\mathbb{R}),$$

so that $(c_0(\mathbb{R}), \|\cdot\|_{l_\infty})$ is the completion of $P(\mathbb{R})$.

To prove that $\overline{l_0(\mathbb{R})} = c_0(\mathbb{R})$ in l_∞ one first notes that any element $\{y_j\}_{j\in\mathbb{N}} \in c_0(\mathbb{R})$ can be approximated by

$$x_1 = (y_1, 0, 0\ldots), \quad x_2 = (y_1, y_2, 0, 0\ldots), \quad x_n = (y_1, y_2, \ldots, y_n, 0, 0\ldots) \in l_0(\mathbb{R}), \qquad n \in \mathbb{N},$$

since

$$\|x_n - \{y_j\}_j\|_{l_\infty} = \sup_{j>n} |y_j| \to 0 \quad \text{as} \quad n \to \infty,$$

by the definition of a limit, and the fact that $\lim_{j\to\infty} y_j = 0$.

The same argument shows that an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}} \in l_0(\mathbb{R})$ cannot converge (with respect to the l_∞-metric) to an element outside of $c_0(\mathbb{R})$, since, for any $x_n \in l_0(\mathbb{R})$, there exists an integer N_n, such that

$$\|x_n - \{y_j\}_j\|_{l_\infty} \ge \sup_{j \ge N_n} |y_j|$$

which may vanish as $n \to \infty$ only if $y_j \to 0$ as $j \to \infty$ (and it vanishes exactly if, up to finitely many terms, $\{x_n\}_n$ is chosen as any subsequence of the sequence described above).