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Department of Mathematical Sciences

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Exam in TMA4145 Linear Methods

Wednesday, December 5, 2012
Time: 09:00 – 13:00
Examination aids: Code D
Grades: Friday, January 4, 2013

No written or handwritten material; calculators Citizen SR-270X (including the *College* version) or Hewlett Packard HP30S. Each correctly solved problem gives 10 points. Within each problem, subproblems give the same amount of points, except for in problems 2, 4 and 5, where the last subproblem gives one point extra. All answers shall be thoroughly motivated (except for in problem 1).

Problem 1 *(Overview)*
For each of the following, indicate whether it is true or false (no proof required).

(i) Any vector space contains a zero element.

(ii) $P_n(\mathbb{R}) \cong \mathbb{R}^n$.

(iii) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable.

(iv) In a normed space, all Cauchy sequences are convergent.

(v) $\{e^{ikx}\}_{k\in\mathbb{Z}}$ defines an orthonormal basis for $L_2((-\pi,\pi),\mathbb{C})$, $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

(vi) $\mathbb{C}^n$ is the dual of $\mathbb{R}^n$.

(vii) A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\ker(T) = \{0\}$.

(viii) The function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, is locally Lipschitz continuous.

(ix) Pythagoras theorem, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \perp y$, holds in inner-product spaces.

(x) The differential operator $1 + (\frac{d}{dx})^2$ is invertible on $C^\infty_{2\pi\text{-per}}(\mathbb{R})$.

Problem 2 *(Finite-dimensional linear transformations)*

a) Find an LU-decomposition of

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}.$$

What is $\ker(A)$, $\operatorname{ran}(A)$?

b) Determine the coordinates $(x_1, x_2, x_3)_A$ of an arbitrary vector $y = (y_1, y_2, y_3)$ in the basis $\{A_1, A_2, A_3\}$ given by the column vectors of A.

c) Perform a Gram–Schmidt orthonormalization of $\{A_1, A_2, A_3\}$.

Problem 3 *(Spectral theory, differential equations)*

a) Express $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ in Jordan normal form TJT^{-1}.

b) Solve the initial-value problem $\dot{x} = Ax$, $x(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Problem 4 *(Normed spaces, minimal distances)*
Let $M \subset X$ be a subset of normed space $(X, \|\cdot\|)$, $x_0 \in X$. In each of the cases listed, calculate $\inf_{y\in M} \|x_0 - y\|$ and determine whether there exists a unique $y_0 \in M$ such that $\|x_0 - y_0\| = \inf_{y\in M} \|x_0 - y\|$. If so, find y_0.

a) $X = \mathbb{R}^3$ with norm $\|x\| = (\sum_{j=1}^3 x_j^2)^{1/2}$, $M = \operatorname{span}\{(1,1,0)\}$, $x_0 = (1,2,3)$.

b) $X = l_\infty(\mathbb{R})$ with norm $\|x\|_{l_\infty} = \sup_{j\in\mathbb{N}} |x_j|$, $M = \{\{x_j\}_{j\in\mathbb{N}} \colon \lim_{j\to\infty} x_j = 0\}$, $x_0 = \{\frac{j}{j+1}\}_{j\in\mathbb{N}}$.

c) $X = L_2((0,1),\mathbb{R})$ with norm $\|x\| = (\int_0^1 |x(t)|^2\,dt)^{1/2}$, $M = \operatorname{span}\{1, t\}$, the function x_0 given by $x_0(t) = t^2$.

Problem 5 *(Hilbert spaces)*
Let $U \in B(H)$ be a unitary operator on a complex Hilbert space H, i.e., a bounded linear operator $H \to H$ such that

$$UU^* = U^*U = \mathrm{Id}.$$

Show that:

a) All eigenvalues of U satisfy $|\lambda| = 1$.

b) If $(U - \lambda)x = 0$, then $(U^* - \overline{\lambda})x = 0$.

c) Eigenspaces corresponding to different eigenvalues of U are orthogonal.

Problem 6 *(Banach spaces, isomorphisms)*
Let $P(\mathbb{R})$ denote the space of all polynomials $p(x) = \sum_k a_k x^k$ (finite sum) with real coefficients $a_0, a_1, \ldots$, equipped with the norm $\|p(x)\| = \max_k |a_k|$.

a) Find a Cauchy sequence in $P(\mathbb{R})$ that does not converge in $P(\mathbb{R})$.

b) Determine the completion of $P(\mathbb{R})$, i.e., find a normed space $\tilde{X}$ and a Banach space X such that $P(\mathbb{R}) \cong \tilde{X} \overset{\text{dense}}{\subset} X$. Here, '$\cong$' indicates an isometric isomorphism, i.e., an isomorphism between normed spaces.