NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4140 Discrete Mathematics**

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Examination time (from–to): 00:00 - 23:59

Permitted examination support material: Permitted examination aids code C: Specified printed and hand-written support material is allowed. A specific basic calculator is allowed. *We specify that it is allowed to bring a stamped yellow A4 sheet with your own handwritten formulas and notes. The calculators allowed for examination aids code C is listed on the NTNU website.*

!!! Note that this is a mock exam for fall 2022, not the real exam !!!

Language: English

Number of pages: 5

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☐	**farger** ☒
skal ha flervalgskjema ☐	

Date Signature

This exam has the following structure:

Exercise 1:	Logic	10 points
Exercise 2:	Sets	10 points
Exercise 3:	Functions	10 points
Exercise 4:	Boolean algebra	10 points
Exercise 5:	Relations	10 points
Exercise 6:	Induction	10 points
Exercise 7:	Combinatorics	10 points
Exercise 8:	Graphs and trees	10 points
Exercise 9:	Number theory	10 points
Exercise 10:	Finite state machines and automata	10 points
Total:		**100 points**

The answer to every problem requires a detailed argument/computation.

Problem 1 **Logic** (10 points)

a. (3 points) Use truth tables to determine which of the following statements are tautologies and which are contradictions (1 point each):

   i) $q \vee (q \to \neg q)$

   ii) $\neg((\neg r \wedge r) \to s)$

   iii) $((t \to s) \to t) \to t$

b. (7 points) Let p, q, r, s, t, u be primitive statements. Provide a step by step verification of the inference

$$\Big((\neg p \vee q) \to r\Big) \wedge \Big(r \to (s \vee t)\Big) \wedge \Big(\neg s \wedge \neg u\Big) \wedge \Big(\neg u \to \neg t\Big) \to p$$

Problem 2 **Sets** (10 points)

a. (4 points) Let X, Y be arbitrary sets. Use the laws of set theory to show that:

$$\text{If} \quad (X \cup Y) \subseteq (X \cap Y) \quad \text{then} \quad X = Y.$$

b. (6 points) Use the laws of set theory to show that for sets X, Y, Z we have that

$$(Y - Z) \subseteq \overline{X} \text{ if and only if } (X \cap Y) \subseteq Z.$$

Problem 3 **Functions** (10 points)

a. (5 points) Let $g : A \to B$ and $f : B \to C$ be two functions. Show that if g and f are both injective, then $f \circ g : A \to C$ is injective.

b. (5 points) Let $f : A \to B$ and $g : B \to C$. Show that if $g \circ f : A \to C$ is surjective, then g is surjective.

Problem 4 **Boolean algebra** (10 points)

a. (1 point) Let $\mathbb{B}$ be a Boolean algebra. Simplify the following Boolean expression

$$(x \cdot x \cdot x \cdot y \cdot y + \overline{x} \cdot y \cdot y) \cdot \overline{(x \cdot x + x \cdot \overline{y} \cdot \overline{y} \cdot \overline{y})}.$$

b. (3 points) Let $\mathbb{B}$ be a Boolean algebra. Show that $x = 0$ if and only if $y = x\overline{y} + \overline{x}y$ for all y.

c. (6 points) Consider the set $X = \{1, 2, 4, 8\}$ and define the binary operations $+ : X \times X \to X$ and $\cdot : X \times X \to X$, where

$$x + y := \operatorname{lcm}(x, y) \quad \text{resp.} \quad x \cdot y := \gcd(x, y)$$

Write down the definitions of least common multiple (lcm) and the greatest common divisor (gcd)). Define the unary operation $^{-} : X \to X$, $x \mapsto \overline{x} := 8/x$. Is $(X, +, \cdot, ^{-}, 1, 8)$ a Boolean algebra? Provide a detailed justification for your answer.

Problem 5 **Relations** (10 points)

a. (4 points) Draw the Hasse diagram for the divisors of 343.

b. (6 points) Recall that a relation R on a set A is antisymmetric if it does not contain any pair of distinct elements of A each of which is related by R to the other. Prove that the relation R defined on the set A is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a, a) \mid a \in A\}$.

Problem 6 **Induction** (10 points)

a. (1 point) Show that for all natural numbers: $4\sum_{k=1}^{n}(k^2 + 2k)(k + 4) = (n^2 + n)(n + 4)(n + 5)$.

b. (3 points) Show that $\sum_{k=1}^{n} k(k!) = (n + 1)! - 1$.

c. (6 points)

   i) (2 points) Show that $n^2 \geq 2n + 1$ for $n > 2$.

   ii) (4 points) Then determine for which natural numbers we have $2^n \geq n^2$.

Problem 7 **Combinatorics** (10 points)

a. (4 points) A string is a palindrome if the string is the same when we read it from the left and from the right, for example: *tacocat*. Let the alphabet be the symbols $\{a, b, c, d, e\}$. How many palindromes of length 11 exists over this alphabet?

b. (6 points) Let L be the language $\{a, b, c\}^*$ consisting of all strings over the alphabet $\{a, b, c\}$. How many

   i) (1 point) strings in L has length 6?

   ii) (2 points) strings in L of length 6 has at least two a's?

   iii) (3 points) strings in L of length 6 are sorted (so that all a's are in front of all b's which are in front of all c's)?

Problem 8 **Graphs and trees** (10 points)

a. (4 points) Let G be a graph with vertices $V = \{1, 2, 3, 4, 5, 6\}$. Draw the graph of the following adjacency matrix, where an index (i, j) in the matrix is 0 if there is no edge from vertex i to vertex j and 1 if there is an edge. Is the graph connected? Is the graph complete?

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

b. (6 points) Does the graph have

i) (1 point) an Euler trail?

ii) (1 point) an Euler circuit?

iii) (2 points) a Hamiltonian path?

iv) (2 points) a Hamiltonian cycle?

Problem 9 **Number theory** (10 points)

a. (4 points) Let $(n, e) = (143, 11)$ be the public key in the RSA cryptosystem. Find the secret key (p, q, d) and decrypt the ciphertext $c = 5$ to find the secret message m.

b. (6 points) Find all integer solutions x to the following system of congruences:

$$\begin{aligned} 3x + 2 &\equiv 3 \pmod 7 \\ x - 4 &\equiv 1 \pmod 5 \\ 5x &\equiv 1 \pmod 9 \end{aligned}$$

F	η		μ	
	a	b	a	b
s_0	s_1	s_0	0	0
s_1	s_0	s_0	1	1

Problem 10 Finite state machines and automata (10 points)

a. (2 points) Draw the transition diagram of the finite state machine F with input $I = \{a, b\}$, output $O = \{0, 1\}$ and states $S = \{s_0, s_1\}$ (initial state is s_0) and the following transition table. What is the output for the input string *abba*?

b. (3 points) Draw the transition diagram of a finite state machine with input and output $O = I = \{0, 1\}$, which outputs 1 when it sees the first 0 in the input string and continues outputting 1 until it sees another 0; thereafter it outputs 0. In all other cases it outputs 0.

c. (5 points) Draw the transition diagram of a finite state automaton with input $I = \{a, b\}$ that accepts strings with an even number of a's.