TMA4140
DISCRETE MATHEMATICS
NTNU, H2022

EXERCISE SET 5

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). To get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations).
To be admitted to the exam in December (15/12/), you must have at least 4+4 set out of the 12 assignments approved. At least 4 of the approved exercises sets must be from the first 6 sets, and at least 4 must be from the last 6 exercise sets.
You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Exercise 1. *Let $P(\mathbb{R})$ be the set of all subsets of $\mathbb{R}$. Define a relation $R \subseteq P(\mathbb{R}) \times P(\mathbb{R})$ by $\langle A, B \rangle \in R$ iff for every $\epsilon > 0$ there exists $x \in A$ and $y \in B$ such that $|x - y| < \epsilon$. What are the properties of R? Transitive, antisymmetric, reflexive, symmetric, irreflexive?*

Exercise 2. *We consider the concept of p-closure of a relation R with respect to a wanted property p (see textbooks RA or LZ for details): expand R into a relation R_p with the desired property p in a minimal way, i.e., adding to R as few new pairs as possible.*
We define for a relation $R \subseteq A \times A$:

a) *The* **reflexive closure** *of R is the smallest relation R_r such that $R \subseteq R_r$ and R_r is reflexive.*
b) *The* **symmetric closure** *of R is the smallest relation R_s such that $R \subseteq R_s$ and R_s is symmetric.*
c) *The* **transitive closure** *of R is the smallest relation R_t such that $R \subseteq R_t$ and R_t is transitive.*

Here, smallest means that if $\tilde{R}$ is any other relation containing the relation R and having the particular property p, then the p-closure $R_P \subseteq \tilde{R}$.

Example: $R = \{(1,5),(2,2),(2,4),(4,1),(4,2)\}$ *on the set* $[5] = \{1,2,3,4,5\}$. *Then we have:*

- *reflexive closure* $R_r = R \cup \{(1,1),(3,3),(4,4),(5,5)\}$.
- *symmetric closure:* $R_s = R \cup \{(5.1),(1,4)\}$
- *transitive closure:* $R_t = R \cup \{(2,1),(4,4),(4,5),(2,5)\}$

(1) *Consider the set* $[5] = \{1,2,3,4,5\}$. *What are the i) reflexive, ii) symmetric , iii) transitive closures of*

$$R := \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(3,4),(4,5)\}?$$

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Exercise 3. *Recall that a function f is a relation from a set A, its domain, to a set B, its co-domain, which associates to an element $a \in A$ its image $f(a) \in B$. In this sense, we can consider k-ary functions as functions with k arguments, by specifying the domain to be the Cartesian product of k sets. Hence, a k-ary function is just a one-argument function with the set $A_1 \times \cdots \times A_k$ as its domain. For notational simplification, we write $f(a_1, \ldots, a_k)$ for $f((a_1, \ldots, a_k))$.*

Let x and y be positive integers. Define the function g recursively: for $x < y$, $g(x, y) = 0$, and for $y \leq x$, $g(x, y) = g(x - y, y) + 1$. Compute explicitly $g(2, 3), g(3, 2), g(23, 6), g(14, 3), g(15, 3)$. Is g injective? Justify your answer.

Exercise 4. *Write the truth table for the following statement:*

$$(p \to (q \vee p)) \wedge ((p \vee q) \to \neg p).$$

Exercise 5. *Use mathematical induction to prove that a set with n elements has 2^n distinct subsets.*

Exercise 6. *Let R be a transitive relation. Let aR^nb mean that there is a sequence of related elements*

$$\langle a, x_1 \rangle \in R, \langle x_1, x_2 \rangle \in R, \ldots, \langle x_{n-1}, b \rangle \in R.$$

Use induction to prove that aRb is a logical consequence of aR^nb.

Exercise 7. *Use induction to prove the equality:*

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{1}{3} n(n+1)(n+2),$$

for all $n \geq 1$.

Exercise 8. *Use induction to prove that the following statement holds for all $n \geq 2$:*

$$(1 - \frac{1}{2})(1 - \frac{1}{3}) \cdots (1 - \frac{1}{n}) = \frac{1}{n}.$$

Exercise 9. *Use induction to prove the inequality:*

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n},$$

for all $n \geq 1$.

Exercise 10. *Let $f : \mathbb{N} \to \mathbb{R}$ be the function defined by:*

$$f(n) = \begin{cases} 1, & n = 0, \\ 3, & n = 1, \\ 14, & n = 2, \\ 6f(n-3) - 11f(n-2) + 6f(n-1), & n \geq 3 \end{cases}.$$

Compute $f(3), f(4), f(5)$. Use induction to prove that:

$$f(n) = \frac{5}{2} - 5 \cdot 2^n + \frac{7}{2} \cdot 3^n.$$