TMA4140
DISCRETE MATHEMATICS
NTNU, H2022

Exercise Set 3

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). To get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations).
To be admitted to the exam in December (15/12/), you must have at least 4+4 set out of the 12 assignments approved. At least 4 of the approved exercises sets must be from the first 6 sets, and at least 4 must be from the last 6 exercise sets.
You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Exercise 1. *Is the following statement a contradiction (write down the truth table)?*

$$((\neg p \wedge q) \wedge (q \wedge r)) \wedge \neg q$$

Use the laws of logic to simplify (give all steps):

$$(\neg p \vee \neg q) \wedge ((r \wedge \neg r) \vee p) \wedge p$$

Solution. This is exam problem 1.5 (1) and (2) from Fall 2020. `https://wiki.math.ntnu.no/_media/tma4140/2020h/exam_tma4140-h2020eng-final-solutions.pdf`

Exercise 2. *Let A, B, C be sets.*

(1) *Prove or disprove that $A = B$ is a logical consequence of $A \cup C = B \cup C$.*
(2) *Prove or disprove that $A = B$ is a logical consequence of $A \cap C = B \cap C$.*

Solution. Counterexamples:

(1) $A = \{1\}, B = \{1, 2\}, C = \{2\}$.
(2) $A = \{1\}, B = \{2\}, C = \{3\}$.

Exercise 3. *Prove or disprove the following statements:*

(1) $p \to (q \wedge r)$ *is equivalent to* $(p \to q) \wedge (p \to r)$.
(2) $p \to (q \vee r)$ *is equivalent to* $(p \to q) \vee (p \to r)$.
(3) $(p \wedge q) \to r$ *is equivalent to* $(p \to r) \wedge (q \to r)$.
(4) $(p \vee q) \to r$ *is equivalent to* $(p \to r) \vee (q \to r)$.
(5) $p \wedge (q \to r)$ *is equivalent to* $(p \wedge q) \to (p \wedge r)$.
(6) $p \vee (q \to r)$ *is equivalent to* $(p \vee q) \to (p \vee r)$.

Solution.

Date: September 14, 2022.

(1) Equivalent. If p is false, then both formulas are true. If p is true, then both q and r has to be true for either formula to be true.
(2) Equivalent. If p is false, then both formulas are true. If p is true, then one of q or r has to be true for either formula to be true.
(3) Equivalent. If r is true, then both formulas are true. If r is false, then both p and q has to be true for either formula to be true.
(4) Equivalent. If r is true, then both formulas are true. If r is false, then one of p or q has to be true for either formula to be true.
(5) Not equivalent. If $p = 0$, then the first formula is false and the second formula is true.
(6) Equivalent. If p is true, then both formulas are true. If p is false, then both formulas has the same value as $q \to r$.

Exercise 4. *Prove or disprove the following statements:*

(1) p *is a logical consequence of* $\{q \vee r, q \to p, r \to p\}$.
(2) $p \to q$ *is a logical consequence of* $\{p \to r, r \to q\}$.

Solution.

(1) If $\{q \vee r, q \to p, r \to p\}$ is true, then either q is true or r is true. If q is true and $q \to p$ is true, then p is true. If r is true and $r \to p$ is true, then p is true. Therefore p is a logical consequence.
(2) If $\{p \to r, r \to q\}$ is true, then either p is false or r is true. If p is false, then $p \to q$ is true. If r is true and $r \to q$ is true, then q is true. If q is true, then $p \to q$ is true. Therefore $p \to q$ is a logical consequence.

Exercise 5. *For the following relations on the set* $\{1, 2, 3\}$*, determine which of the following properties the relations has: reflexive, symmetric, transitive, antisymmetric, irreflexive.*

(1) $R_1 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$
(2) $R_2 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 1, 2\rangle\}$
(3) $R_3 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle\}$
(4) $R_4 = \emptyset$
(5) $R_5 = \{\langle 1, 2\rangle\}$
(6) $R_6 = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle\}$
(7) $R_7 = \{\langle 3, 3\rangle, \langle 3, 2\rangle, \langle 2, 1\rangle\}$

Solution.

(1) Reflexive: Yes, every element in $\{1, 2, 3\}$ is related to itself.
Symmetric: Yes, for every pair $\langle x, y\rangle$ in the relation, the pair $\langle y, x\rangle$ is also in the relation.
Transitive: Yes, for every pairs of pairs of the form $\langle x, y\rangle, \langle y, z\rangle$ in the relation, $\langle x, z\rangle$ is also in the relation.
Antisymmetric: Yes, the only pairs of pairs in the relation of the form $\langle x, y\rangle, \langle y, x\rangle$ satisfy $x = y$.
Irreflexive: No, for example the element 1 is related to itself.
(2) Reflexive: Yes, every element in $\{1, 2, 3\}$ is related to itself.
Symmetric: No, 1 is related to 2 but 2 is not related to 1.
Transitive: Yes, for every pairs of pairs of the form $\langle x, y\rangle, \langle y, z\rangle$ in the relation, $\langle x, z\rangle$ is also

in the relation.
Antisymmetric: Yes, the only pairs of pairs in the relation of the form $\langle x, y\rangle, \langle y, x\rangle$ satisfy $x = y$.
Irreflexive: No, for example the element 1 is related to itself.

(3) Reflexive: Yes, every element in $\{1, 2, 3\}$ is related to itself.
Symmetric: No, 1 is related to 2 but 2 is not related to 1.
Transitive: No, 1 is related to 2 and 2 is related to 3 but 1 is not related to 3.
Antisymmetric: Yes, the only pairs of pairs in the relation of the form $\langle x, y\rangle, \langle y, x\rangle$ satisfy $x = y$.
Irreflexive: No, for example the element 1 is related to itself.
(4) Symmetric, transitive, antisymmetric, irreflexive.
(5) Transitive, antisymmetric, irreflexive.
(6) Symmetric.
(7) Antisymmetric.

Exercise 6. *If d, n are natural numbers, we say that d divides n if there exists a natural number m such that $dm = n$. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$. Write down the relation from A to B defined by that $x \in A$ is related to $y \in B$ iff x divides y.*

Solution.

$$\{\langle 2, 6\rangle, \langle 2, 8\rangle, \langle 2, 10\rangle, \langle 3, 6\rangle, \langle 4, 8\rangle\}$$

Exercise 7. *For the following relations on the set $A = \{1, 2, 3\}$, determine which of the following properties the relations has: reflexive, symmetric, transitive, antisymmetric, irreflexive.*

(1) *The relation $\sim$ on A defined by $x \sim y$ iff $x + y$ is even.*
(2) *The relation $\sim$ on A defined by $x \sim y$ iff $2x + y$ is even.*
(3) *The relation $\sim$ on A defined by $x \sim y$ iff $2x + y$ is odd.*
(4) *The relation $\sim$ on A defined by $x \sim y$ iff $x \cdot y$ is even.*

Solution.

(1) Reflexive, symmetric, transitive.
(2) Transitive, antisymmetric.
(3) Transitive.
(4) Symmetric.

Exercise 8. *Let $X = \{1, 2, 3, 4\}$ and let $P(X)$ be the power set of X. Define the subset $\chi \subset P(X)$ by*

$$\chi = \{Y \subseteq X : 1 \in Y \vee 2 \in Y\}.$$

Let R be the relation on χ defined by that x is related to y iff $x \subseteq y$. Show that R is a partial order. Draw the Hasse diagram. You can read about Hasse diagrams on pages 72-73 in the book Logical Methods.

Solution. This is exam problem 4 from Fall 2018. `https://wiki.math.ntnu.no/_media/tma4140/2018h/tma4140h18lf.pdf`

Exercise 9. *Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define the relation R on A by $\langle (a, b), (c, d) \rangle \in R$ iff a divides c and a divides d. Determine which of the following properties is satisfied by R: Reflexiv, symmetric, antisymmetric, transitive.*

Solution. This is exam problem 4 from Fall 2014. `https://www.math.ntnu.no/emner/TMA4140/2009h/eksamener/h2014-lf.pdf`

Exercise 10. *Let R be the relation on $\mathbb{Z}$ defined by $\langle x, y \rangle \in R$ iff 3 divides $x + 2y$. Determine which of the following properties is satisfied by R: Reflexiv, symmetric, antisymmetric, transitive.*

Solution. This is exam problem 4 from Fall 2013. `https://www.math.ntnu.no/emner/TMA4140/2009h/eksamener/h2013-lf.pdf`