TMA4140
DISCRETE MATHEMATICS
NTNU, H2022

Exercise Set 12

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). To get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations).
To be admitted to the exam in December (15/12/), you must have at least 4+4 set out of the 12 assignments approved. At least 4 of the approved exercises sets must be from the first 6 sets, and at least 4 must be from the last 6 exercise sets.
You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Exercise 1. *In Set 11, the relation between the Narayana and the Catalan numbers was mentioned:*

$$\prod_{k=2}^{n} \frac{n+k}{k} = C_n = \sum_{k=1}^{n} N(n,k). \tag{1}$$

Recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$. *Do you see why the first equality in (1) holds? We would like to now prove the second equality.*

i) First Show that

$$\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1} = \binom{n}{k}^2 \frac{n+1}{(k+1)(n+1-k)}.$$

Solution. $\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1} = \frac{n!^2}{k!^2(n-k)!^2} - \frac{n!}{(k-1)!(n+1-k)!} \cdot \frac{n!}{(k+1)!(n-1-k)!} = \frac{n!^2}{k!^2(n-k)!^2} - \frac{n!^2}{(k-1)!(n+1-k)!(k+1)!(n-1-k)!} = \frac{n!^2[(k-1)!(n+1-k)!(k+1)!(n-1-k)! - (n-k)!^2 k!^2]}{(k)!^2(n-k)!^2(k-1)!(n+1-k)!(k+1)!(n-1-k)!} = \frac{n!^2[(k+1)(n+1-k) - k(n-k)]}{k!(n-k)!(k+1)!(n+1-k)!} = \frac{n!^2(n+1)}{k!^2(n-k)!^2(k+1)(n+1-k)} = \frac{n!^2}{k!^2(n-k)!^2} \cdot \frac{n+1}{(k+1)(n+1-k)} = \binom{n}{k}^2 \frac{n+1}{(k+1)(n+1-k)}$. □

Next, show that

$$N(n+1, k+1) = \binom{n}{k}^2 \frac{n+1}{(k+1)(n+1-k)}.$$

Solution. *See handwritten note.*

We conclude with the identity

$$N(n+1, k+1) = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}. \tag{2}$$

Date: November 8, 2022.

Now, we need two identities that are very similar:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{3}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k-2} = \sum_{k=2}^{n} \binom{n}{k}\binom{n}{k-2} = \binom{2n}{n+2}. \tag{4}$$

Hint: Recall that by definition $\binom{n}{-1} = \binom{n}{-2} = 0$*. Then compute the coefficient of* x^n *in* $(1+x)^{2n}$ *and use the Binomial theorem to conclude in both cases.*

We are now ready to derive the second equality (1) *starting from identity* (2) *and using identities* (3) *and* (4).

Solution. *See handwritten note.*

Exercise 2. *The Lucas numbers are defined recursively*

$$L_0 = 2,\ L_1 = 1,\ \textit{and}\ L_n = L_{n-1} + L_{n-2} \qquad \text{for } n > 1.$$

i) Write down the first eight Lucas numbers.

Solution.

(1) $L_0 = 2$
(2) $L_1 = 1$
(3) $L_2 = 3$
(4) $L_3 = 4$
(5) $L_4 = 7$
(6) $L_5 = 11$
(7) $L_6 = 18$
(8) $L_7 = 29$

ii) Use induction to show that for $n > 0$*:* $\sum_{i=1}^{n} iL_i = 4 - L_{n+3} + nL_{n+2}$.

Solution. *See handwritten note.*

Exercise 3. *Problem* 18.10 *on page* 214 *in Antonsen's textbook (Universitetsforlaget edition).*

Solution. *a) We have 7 letters to choose from for each slot in the string, but the L's and O's are indistinguishable, so we must divide* 7!*, the number of ways we could arrange 7 letters, by the number of ways we can arrange the L's and O's amongst themselves,* 4! *and* 3! *respectively. Then we have* $\frac{7!}{3!4!} = 35$ *possible strings.*

b) We have 7 characters total to choose from but some of them are indistinguishable, as above. Then we must divide by the number of ways we can arrange these indistinguishable characters amongst themselves. This gives $\frac{7!}{2!3!} = 420$ *different strings.*

Exercise 4. *Problem* 19.11 *on page* 222 *in Antonsen's textbook (Universitetsforlaget edition).*

Solution. *a) Verify.*

b) Let $M(n)$ *stand for all morse codes of length* n*. Then it is sufficient to show that* $M(n+2) = M(n) + M(n+1)$ *holds for all* n *since then* $M(n)$ *must be a Fibonacci number for all* n*. This follows from the observation that a morse code of length* $n+2$ *must either start with a short bar, of length one, followed by a code of length* $n+1$*, of which there are* $M(n+1)$*, or a long bar of length*

2 followed by a code of length n, of which there are $M(n)$. Then there are $M(n)+M(n+1)$ morse codes of length $n+2$.

Exercise 5. *Justify your answer for each of the following scenarios:*

i) In how many ways can two different numbers be chosen from $\{1, 2, 3, ..., 50\}$ such that their sum is an even number?

Solution. *In order to get an even sum we need to add two even or two odd numbers. There are $\binom{25}{2}$ ways to choose two even numbers or two odd numbers. Then there are $\binom{25}{2}+\binom{25}{2}=600$ ways total to get an even sum.*

ii) In how many ways can 8 distinct bookers be distributed amongst 3 students (A, B, and C) such that A has 4 books, and B and C have 2 books each?

Solution. *A can choose 4 books from the total 8 $\binom{8}{4}$ different ways. For each of these ways B can then choose 2 books from the remaining 4 $\binom{4}{2}$ different ways. Then C is left with the remaining 2. So we have that there are $\binom{8}{4}\binom{4}{2}=420$ total ways to distribute the books in this manner.*

Problem 2 from August 2018: https://wiki.math.ntnu.no/_media/tma4140/2018h/tma4140_kontinuasjonseksamen_hosten_2018.pdf

Exercise 6. *i) How many binary strings of length 20 exist such that the string contain 6 ones and 14 zeros, and every 1 is followed by a minimum of 2 zeros?*

Solution. *We can think of the string $\alpha = 100$ as a single symbol. Then we create a string with six α and 2 zeros $(14-6\cdot 2=2)$, where the we require that α must appear first. Since the first place is filled we are left with seven places that should be filled with 5 α and 2 zeros. This can be done $\binom{7}{2}=21$ ways.*

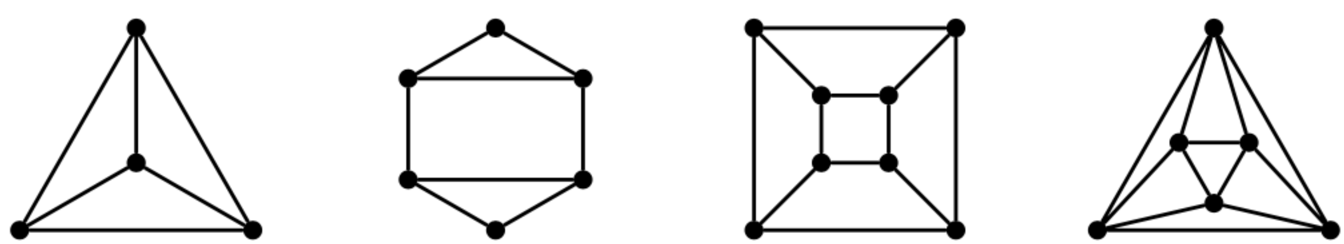
ii) How many ways can you distribute 17 identical objects in four identical boxes, such that each box contains a minimum of 3 objects?

Solution. *We can imagine first that we place 3 objects into each of the boxes. This leaves $17-4\cdot 3=5$ objects remaining to be placed into the four boxes in any manner we choose. Because the objects and boxes are identical we don't care about position only the number of objects in each box. There are only 6 possible ways to do this:* $(5,0,0,0),(4,1,0,0),(3,2,0,0),(3,1,1,0),(2,2,1,0),(2,1,1,1)$.

Problem 3 from August 2022: https://wiki.math.ntnu.no/_media/tma4140/2021h/tma4140_2022k_nb.pdf

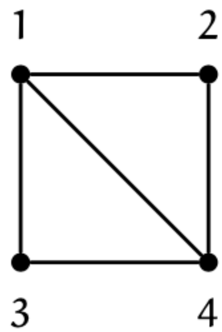
Exercise 7. *Problem* 21.5 *on page* 241 *in Antonsen's textbook (Universitetsforlaget edition).*

Solution.



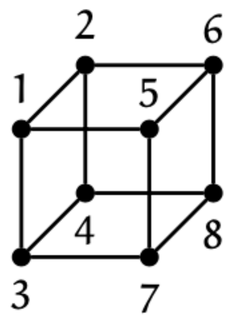
Exercise 8. *Problem* 22.2 *on page* 251 *in Antonsen's textbook (Universitetsforlaget edition).*

Solution.



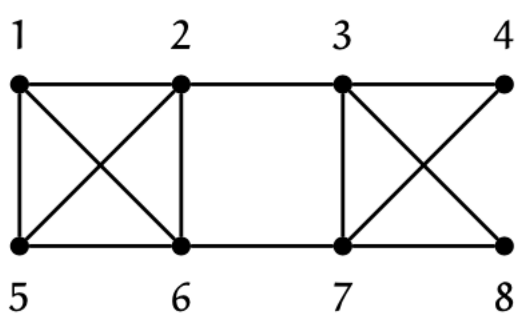
a) This graph does no contain an Euler circuit. We know this because nodes 1 and 4 have odd degrees. However, this graph does contain an Euler path which starts and ends at the nodes with odd degrees

b) A Hamilton cycle is 12431.



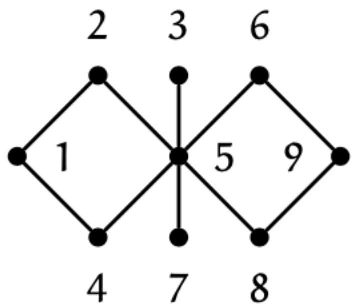
a)This graph has no Euler path and therefore no Euler circuit. This is because there are more than two nodes of odd degree.

b) A Hamilton cycle is 126578431.



a) This graph has exactly two nodes, 1 and 5, of odd degree, and an Euler path is 12347387652615.

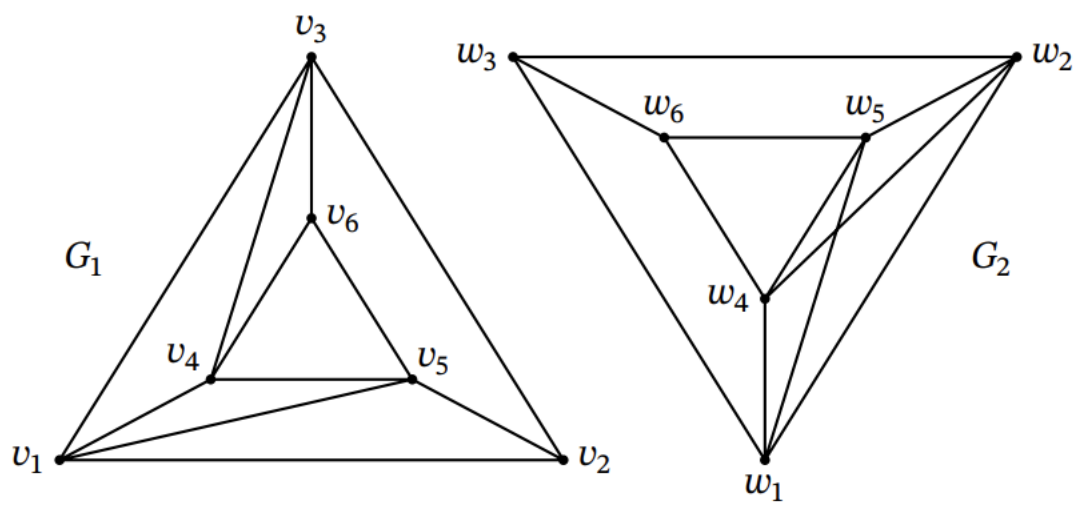
b) There is no Hamilton cycle but there is a Hamilton path: 43215678.



a) An Euler path is 35214569847, but there is no Euler circuit.

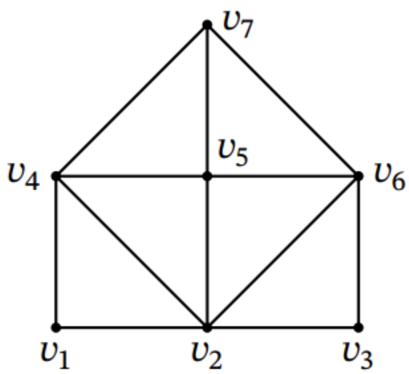
b) The graph has no Hamiltonian path because any path containing nodes 3 and 7 must contain node 5 at least twice.

Exercise 9. *a) Determine if the two graphs in the below figure are isomorphic or not. Justify your answer.*

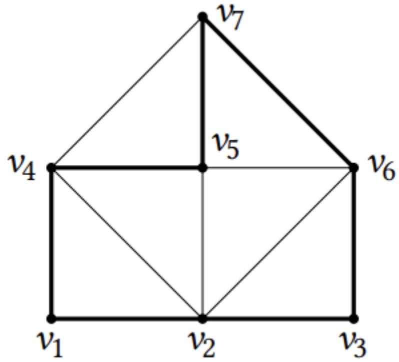


Solution. *The graphs are not isomorphic. Both graphs have two nodes of degree 3 but the two nodes of degree 3 in G_2 are neighbors, connected by an edge, whereas the nodes in G_1 are not. You can also determine that the graphs are not isomorphic by noting that G_2 contains a copy of the complete graph K_4 while G_1 does not.*

b) Determine if the following graph has a Hamilton cycle. Justify your answer.



Solution. *The graph has a Hamilton cycle. See below.*



Problem 5 from August 2022: `https://wiki.math.ntnu.no/_media/tma4140/2021h/tma4140_2022k_nb.pdf`

Exercise 10. *Let the simple undirected graph G with the nodes $\{a_1, a_2, a_3, ..., a_8\}$ be given by the adjacency matrix (with the nodes in the given order):*

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and let the simple undirected graph G' with nodes $\{b_1, b_2, b_3, ..., b_8\}$ given by the adjacency matrix (with the nodes in the given order):

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Determine whether G and G' are isomorphic or not. Justify your answer.

Solution. *The two graphs are not isomorphic. If we color the even numbered nodes from G' with one color and the odd nodes with another we can see that G' is bipartite, however in G we have a triangle, then G is not bipartite and the graphs can't be isomorphic.*

Problem 4 from December 2019: `https://wiki.math.ntnu.no/_media/tma4140/2019h/ntnueksamen-tma4140h19bm.pdf`

Ex 1

ii)

$$N(n+1,k+1) \overset{?}{=} \binom{n}{k}^2 \frac{n+1}{(k+1)(n+1-k)}$$

Recall:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

Then:

$$N(n+1,k+1) = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} = \frac{1}{n+1}\cdot\frac{(n+1)!}{(k+1)!(n-k)!}\cdot\frac{(n+1)!}{k!(n+1-k)!} =$$

$$\frac{1}{n+1}\cdot\frac{(n+1)!}{(k+1)!(n-k)!}\cdot\frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1)!^2}{(k+1)!(n-k)!\,k!(n+1-k)!(n+1)} = \frac{\cancel{(n+1)}(n+1)\,n!^2}{\cancel{(n+1)}(k+1)!(n-k)!\,k!\,(n+1-k)!}$$

$$= \frac{(n+1)\,n!^2}{(k+1)(n+1-k)(n-k)!^2\,k!^2} = \frac{n+1}{(k+1)(n+1-k)}\cdot\frac{n!^2}{(n-k)!^2\,k!^2} = \frac{n+1}{(k+1)(n+1-k)}\cdot\binom{n}{k}^2. \checkmark$$

Ex2

ii) $n=1$ $\quad \sum_{i=1}^{1} iL_i = 4 - L_{n+3} + nL_{n+2}$

$$1 \cdot L_1 = 4 - L_4 + 1 \cdot L_3$$
$$1 = 4 - 7 + 1 \cdot 4$$
$$1 = 1 \checkmark$$

Assume true up to $n=K$: $\sum_{i=1}^{K} iL_i = 4 - L_{K+3} + K \cdot L_{K+2}$

Show this is true for $n=K+1$: $\sum_{i=1}^{K+1} iL_i = 4 - \underbrace{L_{(K+1)+3}}_{L_{K+4}} + (K+1)\underbrace{L_{(K+1)+2}}_{L_{K+3}}$

$$\sum_{i=1}^{K+1} iL_i = \underbrace{1 \cdot L_1 + 2L_2 + \ldots + KL_K}_{= \sum_{i=1}^{K} iL_i = 4 - L_{K+3} + K \cdot L_{K+2}} + (K+1)L_{K+1} = 4 - L_{K+3} + K \cdot L_{K+2} + (K+1)L_{K+1}$$

$$= 4 - L_{K+3} - L_{K+2} + K \cdot L_{K+2} + L_{K+2} + (K+1)L_{K+1} = 4 - (L_{K+3} + L_{K+2}) + (K+1)L_{K+2} + (K+1)L_{K+1}$$

$$= 4 - (L_{K+4}) + (K+1)(L_{K+2} + L_{K+1}) = 4 - L_{K+4} + (K+1)L_{K3}. \checkmark$$

$$\frac{1}{n+1}\binom{2n}{n} = C_n \overset{?}{=} \sum_{k=1}^{n} N(n,k) = \sum_{k=1}^{n} \left((n-1)+1, (k-1)+1\right) \overset{(2)}{=} \sum_{k=1}^{n} \binom{n-1}{k-1}^2 - \binom{n-1}{k-2}\binom{n-1}{k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k}^2 - \binom{n-1}{k-1}\binom{n-1}{k+1} = \sum_{k=0}^{n-1}\binom{n-1}{k}^2 - \sum_{k=0}^{n-1}\binom{n-1}{k-1}\binom{n-1}{k+1}$$

$$\overset{(3)}{=} \binom{2(n-1)}{n-1} - \overset{(4)}{\binom{2(n-1)}{(n-1)+2}} = \binom{2n-2}{n-1} - \binom{2n-2}{n+1}$$

$$\frac{(2n-2)!}{(n-1)!(2n-2-n+1)!} - \frac{(2n-2)!}{(n+1)!(2n-2-n-1)!} = \frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-2)!}{(n+1)!(n-3)!}$$

$$= \frac{(2n-2)!\left[(n+1)!(n-3)! - (n-1)!^2\right]}{(n-1)!(n-1)!(n+1)!(n-3)!} = \frac{(2n-2)!\cancel{(n-1)!}\cancel{(n-3)!}\left[(n+1)(n)-(n-1)(n-2)\right]}{\cancel{(n-1)!}(n-1)!(n+1)!\cancel{(n-3)!}}$$

$$= \frac{(2n-2)!\,[\cancel{n^2}+n-(\cancel{n^2}-3n+2)]}{(n-1)!(n+1)!} = \frac{2(2n-1)(2n-2)!}{(n+1)(n-1)!\,n!} = \frac{2(2n-1)!}{(n+1)(n-1)!\,n!}\cdot\frac{n}{n}$$

$4n-2$

$$= \frac{2n(2n-1)!}{(n+1)\,n!\,n!} = \frac{1}{n+1}\cdot\frac{(2n)!}{n!\,n!} = C_n$$