TMA4140
DISCRETE MATHEMATICS
NTNU, H2022

Exercise Set 11

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). To get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations).
To be admitted to the exam in December (15/12/), you must have at least 4+4 set out of the 12 assignments approved. At least 4 of the approved exercises sets must be from the first 6 sets, and at least 4 must be from the last 6 exercise sets.
You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Exercise 1. *Let* $X := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43\}$ *be ordered by divisibility. Find the maximal and minimal elements of* X*.*

Solution. All elements $x \in X$ are both maximal and minimal. Because all elements are primes and divide only themselves it is true for all that both aRx and xRa imply that $a = x$.

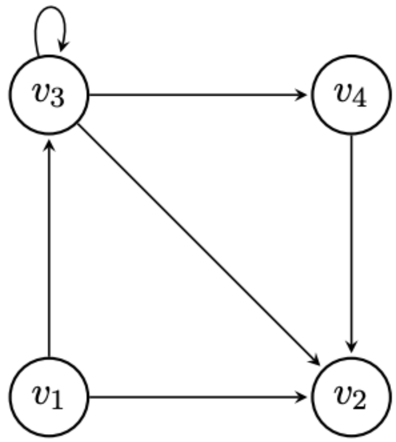
Exercise 2. *Let* R *be a relation with matrix representation*

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Draw the corresponding directed graph.

Solution.

Date: November 1, 2022.



Exercise 3. (1) *Recall that a function f from set A to set B, written $f : A \to B$, is a special kind of relation, that is, a set of ordered pairs (i.e., a subset of the Cartesian product, $f \subseteq A \times B$), which has a specific rule: every element in A must have exactly one B-value.*

Consider the two sets $[3] = \{1, 2, 3\}$ and $\{0, 1\}$.

i) Write down all functions from $[3]$ to $\{0, 1\}$ in the format of ordered-pairs.

ii) State for each of these functions whether it is injective, surjective, or both.

Now, suppose that n is a positive integer (See Problem 19.4 in chapter 19 in RA.).

iii) How many functions are there from the set $[n]$ to $\{0, 1\}$?

iv) How many functions are there from the set $[n]$ to $\{0, 1, 2\}$?

v) How many functions are there from the set $[n]$ to $\{0, 1, 2, 3\}$?

vi) For each case above, how many of these are injective?

(2) *Consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = x^2 - 3x + 2$ (a so-called quadratic polynomial).*

i) Is f injective, surjective, or both?

ii) Is $f : \mathbb{R} \to [-0.25, \infty[$, $x \mapsto f(x) = x^2 - 3x + 2$, injective, surjective, or both?

ii) Determine the domain $X \subseteq \mathbb{R}$ and the co-domain $Y \subseteq \mathbb{R}$ such that $f : X \to Y$, $x \mapsto f(x) = x^2 - 3x + 2$, is bijective.

Solution.

(1) i)

(a) $(1, 0), (2, 0), (3, 0)$
(b) $(1, 1), (2, 0), (3, 0)$
(c) $(1, 0), (2, 1), (3, 0)$
(d) $(1, 0), (2.0), (3, 1)$
(e) $(1, 1), (2, 1), (3, 0)$
(f) $(1, 1), (2, 0), (3, 1)$
(g) $(1, 0), (2, 1), (3, 1)$
(h) $(1, 1), (2, 1), (3, 1)$

ii) 2-7 are surjective. None are injective and then obviously none can be both.

iii) There are 2^n such functions. For each element in $[n]$ we have 2 possible ordered pairs, then we choose from these 2 for each of the elements in $[n]$ which gives $2 \cdot 2 \cdot ... \cdot 2 = 2^n$.

iv) There are 3^n such choices. We now have 3 possible ordered pairs for each element in $[n]$. We then choose from these possibilities for each of the n elements in $[n]$. Then the total number of functions is given by $3 \cdot 3 \cdot ... \cdot 3 = 3^n$

v) There are 4^n such choices given by the same reasoning as above.

vi) Note that in order for a function to be injective the number of elements in the domain must be less than or equal to the number of elements in the range. If this is true then we have: ${}^2P_n$, ${}^3P_n$, and ${}^4P_n$ respectively.

(2) i) Neither injective nor surjective.

ii) Surjective.

iii) $f : [\frac{3}{2}, \infty] \to [-\frac{1}{4}, \infty]$

Exercise 4. *Show that for all positive integers m, n we have*

$$i)\ n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}.$$

$$ii)\ \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

iii) We define the Narayana numbers:

$$N(0,0) = 1, \qquad N(n,0) = 0,\ n > 0, \qquad N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},\ n \geq k \geq 1.$$

Show that $N(n, n+1-k) = N(n,k)$. Compute the numbers $N(4,k)$, for $k = 1, \dots, 4$.

Note: Narayana numbers count the number of expressions with n correctly matched pairs of parentheses with k distinct nestings[1]*. For instance, $N(4,2) = 6$, which is the number of 4 correctly matched pairs of parentheses with 2 distinct nestings:*

$$(()(())) \qquad ((()())) \qquad ((())()) \qquad ()((())) \qquad (())(()) \qquad ((()))().$$

It also gives the number Dyck paths (staircase walks)[2] *of length n with exactly k peaks. Recalling the notion of (inductively defined planar) binary trees, we see that Narayana numbers are therefore intimately related to the famous Catalan numbers*

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!},$$

$C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42, \dots$, which from probably the most important sequence of numbers in combinatorics[3] *Indeed, we have that*

$$C_n = \sum_{k=1}^{n} N(n,k).$$

Check this statement for $n = 4$.

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[1] See M. Bremner's sl;ides `https://math.usask.ca/~bremner/research/publications/Beamer-Narayana-9.pdf`.

[2] Wikipedia: `https://en.wikipedia.org/wiki/Catalan_number`

[3] See R. Stanley's (MIT, USA): slides `https://math.mit.edu/~rstan/transparencies/china.pdf`.

See also Stanley's Exercises on Catalan and Related Numbers `https://math.mit.edu/~rstan/ec/catalan.pdf` and I. Pak's (UCLA, USA) website `https://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm`.

Stanley has written a book on Catalan numbers `https://www.cambridge.org/core/books/catalan-numbers/5441FB5B09E9C01185834D9CBB9DFAD9`.

Solution.
i) $n\binom{m+n}{m} = n\binom{m+n}{(m+n)-m} = n\binom{m+n}{n} =$
$n\frac{(m+n)!}{n!(m+n-n)!} = n\frac{(m+n)!}{n!m!} = \frac{(m+n)!}{(n-1)!m!} =$
$\frac{(m+1)(m+n)!}{(n-1)!(m+1)!} = \frac{(m+1)(m+n)!}{((m+n)-(m+1))!(m+1)!} =$
$(m+1)\binom{m+n}{m+1}$.
ii) $\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} =$
$\frac{(2n)!(n-1)!(n+1)!-(2n)!n!n!}{n!n!(n-1)!(n+1)!} = \frac{[(2n)!n!(n-1)!][(n+1)-n]}{n!n!(n-1)!(n+1)!} =$
$\frac{(2n)!}{n!(n+1!)} = \frac{(2n)!}{(n+1)n!n!} =$
$\frac{1}{n+1}\binom{2n}{n}$.
iii) $N(n, n+1-k) = \frac{1}{n}\binom{n}{n+1-k}\binom{n}{(n+1-k)-1} =$
$\frac{1}{n}\binom{n}{n+1-k}\binom{n}{(n-k)} = \frac{1}{n}\binom{n}{n+1-k}\binom{n}{(k)} =$
$\frac{1}{n}\binom{n}{n-(k-1)}\binom{n}{(k)} = \frac{1}{n}\binom{n}{k-1}\binom{n}{(k)} =$
$\frac{1}{n}\binom{n}{k}\binom{n}{k-1} = N(n,k)$.
$N(4,k)$, for $k = 1, \dots, 4$: $N(4,1) = 1$, $N(4,2) = 6$, $N(4,3) = 6$, $N(4,4) = 1$.

Exercise 5. *Problem 2 from august 2019:* `https://wiki.math.ntnu.no/_media/tma4140/2019h/ntnueksamen-tma4140k19bm.pdf`

Solution. a) Using the formula for binomal coefficients we get $\binom{10}{6}(2^4)(-3^6) = 2,449,440$ as the coefficient for x^4y^6.

b)Recall from exercise 3 that there are 8^n total functions and ${}^8P_4$ injective functions.

Exercise 6. *Problem 4 from august 207:* `https://www.math.ntnu.no/emner/TMA4140/2009h/eksamener/k2017.pdf`

Solution. a) We will consider the shelves to be dividers of the set of n indistinguishable books. Then there are actually $k-1$ dividers. We can think of this as having $n+k-1$ items and needing to choose $k-1$ of them to act as points of separation. Then we have that there are $\binom{n+k-1}{k-1}$ ways to arrange the books in this case.

b) This case is similar to the previous case but now we must also consider how many ways we can arrange the books. We still have $\binom{n+k-1}{k-1}$ ways to separate the books amongst the shelves, but now we also have $n!$ ways to arrange the books. Then the total number of ways to arrange the books on the shelves for this case is $n! \cdot \binom{n+k-1}{k-1}$.

Exercise 7. *Problem 18.1 in chapter 18 in RA.*

Solution. a) $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.

b) $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$

c) Assuming that $n \geq 8$ we have $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-7)$.

d) We have to choose 2 squares from 8 squares to place green or blue in and then a remaining 6 to squares to choose 2 from for the placement of the other color, this gives: $\binom{8}{2}\binom{6}{2} = 420$.

e)We have a total of $\binom{8}{2}$ ways to place 2 blue squares amongst 8 total. There are 7 placements for which the blue squares are next to one another. Then we have that $\binom{8}{2} - 7 = 21$ is the total number of ways to place 2 blue squares that are not adjacent to one another.

Exercise 8. *Problem 18.5 in chapter 18 in RA.*

Solution.a) 120

b) 720

c) 6

d) 9

e) 20

f) 336

g) 1

h) 1

i) 10

j) 120

k) 495

l) 1365

Exercise 9. *Problem 18.8 in chapter 18 in RA.*

Solution. We have 26 choices for the first character in the password. Then for the remaining characters, with no restrictions we would have 36^7 choices. However we want to exclude those passwords which consist only of letters. of which there are 26^7. Then we have $26 \cdot (36^7 - 26^7)$ total possible passwords which start with a letter and contain at least one digit.

Exercise 10. *Problem 19.10 in chapter 19 in RA.*

Solution. a) Length 1: 0, 1, and 2.

Length 2: 00, 01, 02, 11, 12, 22.

Length 3: 000, 001, 002, 011, 012, 022, 111, 112, 122 and 222.

b) Length 4: 0000, 0001, 0002, 0011, 0012, 0022, 0111, 0112, 0122, 0222, 1111, 1112, 1122, 1222 and 2222.

Length 5: 00000, 00001, 00002, 00011, 00012, 00022, 00111, 00112, 00122, 00222, 01111, 01112, 01122, 01222, 02222, 11111, 11112, 11122, 11222, 12222 and 22222.

c) A sorted string with length m will first consist of some number of 0's, then some numbers of 1's, then some number of 2's. Similar to 6a) we can consider this to be a problem in which we simply need to choose where to place the division between the different values. Then we need 2 dividers which means we have in total $m + 2$ elements of which we need to choose 2 to act as a divider where the values of the characters will increase. So we have $\binom{m+2}{2}$.

d) The number of dividers is dictated by the number of elements. we need one less divider than number of elements, $n - 1$. Then we have $\binom{m+n-1}{n-1}$ or, equivalently, $\binom{m+n-1}{m}$ strings given n elements in the alphabet and a string of length m.