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This exam in TMA4140 Discrete Mathematics has the following structure:

Problem 1:	Logic	10 points
Problem 2:	Sets	10 points
Problem 3:	Functions	10 points
Problem 4:	Boolean algebra	10 points
Problem 5:	Relations	10 points
Problem 6:	Induction	10 points
Problem 7:	Combinatorics	10 points
Problem 8:	Graphs and trees	10 points
Problem 9:	Number theory	10 points
Problem 10:	Finite state machines and automata	10 points
Total:		**100 points**

The answer to every problem requires a detailed argument/computation.

Problem 1 **Logic** (10 points)

a. (3 points) Use a truth table to decide whether $[(\neg p \to q) \wedge (p \to q)] \to q$ is a tautology or a contradiction.

b. (3 points) Write the following proposition in conjunctive normal form (CNF)

$$p \to (q \wedge \neg r).$$

c. (4 points) Use 1st-order logic to write down the definition of surjectivity (onto) for a function $f : X \to Y$ and then form the negation of the statement so that no negation sign is to the left of a quantifier. Express the meaning of the negation in simple words.

Problem 2 **Sets** (10 points)

a. (3 points) What is the power set of $A := \{\{a, b\}, \{c, d, e\}, \{f\}\}$?

b. (7 points) Let A, B, C be sets. The symmetric difference is defined by $A \triangle B := (A \backslash B) \cup (B \backslash A)$. Use an element argument to show that $A \triangle B = (A \cup B) \backslash (A \cap B)$.

Problem 3 Functions (10 points)

a. (3 points) Consider sets $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $C = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and the functions

$$f_1 = \{(a, 2), (b, 3), (c, 1)\} \qquad \text{and} \qquad f_2 = \{(1, \alpha_2), (2, \alpha_2), (3, \alpha_4), (4, \alpha_1)\}.$$

Write down the composition $f_2 \circ f_1$ as a set of ordered pairs and draw its arrow diagram.

b. (7 points) Let m be an odd integer, that is, $m = 2l + 1$ for some integer l. Show that

$$\lfloor \frac{m^2}{4} \rfloor = \left(\frac{m-1}{2}\right)\left(\frac{m+1}{2}\right).$$

Problem 4 Boolean algebra (10 points)

Consider the notions of least common multiple (lcm) and the greatest common divisor (gcd). Let $S = \{1, 2, 3, 6\}$ and define the binary operations $x + y := \mathrm{lcm}(x, y)$ and $x \cdot y := \gcd(x, y)$ and the unary operation $\overline{x} := 6/x$. Show that $(S, +, \cdot, ^{-}, 1, 6)$ is a Boolean algebra.

Problem 5 Relation (10 points)

a. (3 points) Draw all possible distinct Hasse diagrams for a poset with 3 elements.

b. (7 points) Draw the Hasse diagram for the poset formed by the divisors of 42.

Problem 6 Induction (10 points)

For a positive integer n, define the triangular number $t_n = 1 + 2 + \cdots + n$. Use induction to show that $\sum_{i=1}^{n} t_i = \frac{n+2}{3} t_n$.

Hint: Recall that Carl Friedrich Gauss computed the number t_{100} as $(100 \cdot 101)/2$.

Problem 7 **Combinatorics** (10 points)

a. (4 points) What is the coefficient in front of the term x^6y^3 in the polynomial $(3x+4y)^9$?

b. (6 points) We have a bin with 4 balls in different colors: red, green, brown and blue. In how many ways can we

   i) (1 point) sample different outcomes?

   ii) (2 points) sample different outcomes when we do not sample the red ball last?

   iii) (3 points) sample 6 balls from the bin when we do not care about the order we sample, and we put each ball back in the bin after it is sampled?

Problem 8 **Graphs & trees** (10 points)

a. (4 points) Let G be a graph with vertices $V = \{1, 2, 3, 4, 5\}$. Draw the graph of the following adjacency matrix, where an index (i, j) in the matrix is 0 if there is no edge from vertex i to vertex j and 1 if there is an edge. Is G directed or undirected? Is G a simple graph or a multigraph?

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. (6 points) Does the graph have

   i) (1 point) an Euler trail?

   ii) (1 point) an Euler circuit?

   iii) (2 points) a Hamiltonian path?

   iv) (2 points) a Hamiltonian cycle?

Problem 9 **Number theory** (10 points)

a. (4 points) What does Euler's theorem say? Compute $134^{36362} \pmod{675}$.

b. (6 points) Find all integer solutions x to the following system of congruences:

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 4 \pmod{7} \\ x &\equiv 5 \pmod{8} \end{aligned}$$

Problem 10 **Finite state machines & automata** (10 points)

a. (4 points) Draw the transition diagram of the finite state machine F with input $I = \{a, b, c\}$, output $O = \{0, 1\}$ and states $S = \{s_0, s_1, s_2\}$ (initial state is s_0) and the following transition table. What is the output for the input string *aabbcc*?

F	η			μ		
	a	b	c	a	b	c
s_0	s_0	s_1	s_2	0	1	0
s_1	s_1	s_1	s_0	1	1	1
s_2	s_2	s_1	s_0	1	0	0

b. (6 points) Draw the transition diagram of an automaton with five states and input $I = \{a, b\}$ accepting strings containing the sub-string: *abba*.