NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4140 Discrete Mathematics**

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Permitted examination support material: Permitted examination aids code C: Specified printed and hand-written support material is allowed. A specific basic calculator is allowed. *We specify that it is allowed to bring a stamped yellow A4 sheet with your own handwritten formulas and notes. The calculators allowed for examination aids code C is listed on the NTNU website.*

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Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☐	**farger** ☒
skal ha flervalgskjema ☐	

Date Signature

This exam has the following structure:

Exercise 1:	Logic	10 points
Exercise 2:	Sets	10 points
Exercise 3:	Functions	10 points
Exercise 4:	Boolean algebra	10 points
Exercise 5:	Relations	10 points
Exercise 6:	Induction	10 points
Exercise 7:	Combinatorics	10 points
Exercise 8:	Graphs and trees	10 points
Exercise 9:	Number theory	10 points
Exercise 10:	Finite state machines and automata	10 points
Total:		**100 points**

The answer to every problem requires a detailed argument/computation.

Problem 1 **Logic** (10 points)

a. (3 points) Use a truth table to decide whether $(p \to q) \to [(p \to \neg q) \to \neg p]$ is a tautology or a contradiction.

b. (7 points) Write the following proposition in conjunctive normal form (CNF)

$$[p \wedge (q \vee r)] \to (q \wedge r).$$

Solution.

a. Tautology.

p	q	$p \to q$	$p \to \neg q$	$(p \to \neg q) \to \neg p$	$(p \to q) \to [(p \to \neg q) \to \neg p]$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	1	0	1	1

b. Conjunctive normal form means that the result should be of the form $X \wedge Y$.

$$\begin{aligned}
[p \wedge (q \vee r)] \to (q \wedge r) &\iff \neg[p \wedge (q \vee r)] \vee (q \wedge r) \\
&\iff [\neg p \vee \neg(q \vee r)] \vee (q \wedge r) \\
&\iff \neg p \vee (\neg q \wedge \neg r) \vee (q \wedge r) \\
&\iff [(\neg p \vee \neg q) \wedge (\neg p \vee \neg r)] \vee (q \wedge r) \\
&\iff ([(\neg p \vee \neg q) \wedge (\neg p \vee \neg r)] \vee q) \wedge ([(\neg p \vee \neg q) \wedge (\neg p \vee \neg r)] \vee r) \\
&\iff (\neg p \vee \neg q \vee q) \wedge (\neg p \vee \neg r \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg r \vee r) \\
&\iff T \wedge (\neg p \vee \neg r \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge T \\
&\iff (\neg p \vee \neg r \vee q) \wedge (\neg p \vee \neg q \vee r).
\end{aligned}$$

Problem 2 **Sets** (10 points)

a. (5 points) Show that for power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$ the following is true

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

b. (5 points) Show that for sets A, B, C then

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Solution.

a. Let $x \in \mathcal{P}(A \cap B)$, then x is a subset of $A \cap B$. Hence x is a subset of A and x is a subset of B. Then $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, so $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Let $y \in \mathcal{P}(A) \cap \mathcal{P}(B)$, then $y \in \mathcal{P}(A)$ and $y \in \mathcal{P}(B)$. So y is a subset of A and y is a subset of B. Then y is a subset of $A \cap B$, so $y \in \mathcal{P}(A \cap B)$.

b. Let $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$. Hence $y \in B$ and $y \in C$. So $(x, y) \in A \times B$ and $(x, y) \in A \times C$, therefore $(x, y) \in (A \times B) \cap (A \times C)$. Let $(a, b) \in (A \times B) \cap (A \times C)$, then $(a, b) \in (A \times B)$ and $(a, b) \in (A \times C)$. So $a \in A$ and $b \in B$ and $b \in C$. Hence $b \in B \cap C$. Therefore $(a, b) \in A \times (B \cap C)$.

Problem 3 Functions (10 points)

a. (2 points) Prove or disprove that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

b. (8 points) Consider the non-empty sets A, B and the functions $f : A \to B$ and $g : B \to A$. Show that f is injective if and only if $g \circ f = \mathrm{id}_A$, that is, f has a left inverse g.

Solution.

a. Counter-example: $x = \frac{1}{2}, y = \frac{1}{2}, \lfloor x + y \rfloor = 1, \lfloor x \rfloor + \lfloor y \rfloor = 0$.

b. Suppose that f has left inverse g. If f is not injective, there exists x, y distinct such that $f(x) = f(y)$, but then $x = g(f(x)) = g(f(y)) = y$ is a contradiction, hence left inverse implies injective. Suppose that f is injective, then it is well-defined to consider $g(y) = \{x : f(x) = y\}$ as a map from the image of f to A, because this set has one unique element for each y. The value of g on elements outside of the image of f does not matter.

Problem 4 Boolean algebra (10 points)

Show that the following statements in Boolean Algebra are logically equivalent.

i) $xy = x$

ii) $x + y = y$

iii) $x\overline{y} = 0$

iv) $\overline{x} + y = 1$

Hint: Try to show that i) implies ii) implies iii) implies iv) implies i).

Solution.

i) $\rightarrow$ ii)

$$\begin{aligned} x + y =& xy + y \\ =& (x+1)y \\ =& 1y = y. \end{aligned}$$

ii) $\rightarrow$ iii)

$$\begin{aligned} x\overline{y} =& x(\overline{x+y}) \\ =& x(\overline{x} \cdot \overline{y}) \\ =& (x \cdot \overline{x})\overline{y} \\ =& 0 \cdot \overline{y} = 0. \end{aligned}$$

iii) $\rightarrow$ iv)

$$\begin{aligned} \overline{x} + y =& \overline{x \cdot \overline{y}} \\ =& \overline{0} = 1. \end{aligned}$$

iv) $\rightarrow$ i)

$$\begin{aligned} xy =& xy + x\overline{x} \\ =& x(\overline{x} + y) \\ =& x1 = x. \end{aligned}$$

Problem 5 Relations (10 points)

a. (4 points) Consider the set $A = \{1, 2, 3, 4\}$ and the relation $R := \{(2, 4), (4, 3), (4, 4)\}$.

   i) (1 point) What is the symmetric closure of the relation R?

   ii) (3 points) What is the reflexive symmetric transitive closure of R?

b. (6 points) Show that R is symmetric if and only if $R \cap R^{-1} = R = R^{-1}$.

Solution.

a. i) $R \cup \{(4,2),(3,4)\}$.

ii) $R \cup \{(1,1),(2,2),(3,3),(4,2),(3,4),(2,3),(3,2)\}$.

b. R is symmetric if and only if $(a,b) \in R$ is equivalent to $(b,a) \in R$. The inverse relation R^{-1} is defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$. So if $(a,b) \in R$, then $(b,a) \in R$ and $(a,b) \in R^{-1}$. Similarly, if $(a,b) \in R^{-1}$, then $(b,a) \in R$ so $(a,b) \in R$. Therefore $R = R^{-1}$ whenever R is symmetric.

Suppose that $R = R^{-1}$ and that $(a,b) \in R$. Then $(b,a) \in R^{-1}$, so $(b,a) \in R$. Therefore R is symmetric. We conclude that the claim holds.

Problem 6 Induction (10 points)

a. (4 points) Show that for non-negative integers n

$$\sum_{i=1}^{n} 3^i = \frac{1}{2}(3^{n+1} - 3).$$

b. (6 points) For positive integer n, consider the sum

$$F(n) = \sum_{i=1}^{n} \frac{1}{i^2 + i}.$$

Determine and prove a formula for $F(n)$.

Hint: try the cases $n = 1, 2, 3$ to deduce a conjectural formula.

Solution.

a. Base case: $3^1 = 3$ and $\frac{1}{2}(3^2 - 3) = \frac{6}{2} = 3$.
Inductive step: Suppose that $\sum_{i=1}^{n} 3^i = \frac{1}{2}(3^{n+1} - 3)$, need to show $\sum_{i=1}^{n+1} 3^i = \frac{1}{2}(3^{n+2} - 3)$:

$$\begin{aligned}\sum_{i=1}^{n+1} 3^i =& \frac{1}{2}(3^{n+1} - 3) + 3^{n+1} \\ =& \frac{3}{2}3^{n+1} - \frac{1}{2}3 \\ =& \frac{1}{2}3^{n+2} - \frac{1}{2}3 \\ =& \frac{1}{2}(3^{n+2} - 3).\end{aligned}$$

b. $F(1) = \frac{1}{2}, F(2) = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}, F(3) = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$. Guess $F(n) = \frac{n}{n+1}$.
Base step: See above.
Inductive step: Suppose that $F(n) = \frac{n}{n+1}$, need to show that $F(n+1) = \frac{n+1}{n+2}$:

$$\begin{aligned}F(n+1) =& \sum_{i=1}^{n+1} \frac{1}{i^2 + i} \\ =& \frac{n}{n+1} + \frac{1}{(n+1)^2 + n + 1} \\ =& \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ =& \frac{n(n+2)+1}{(n+1)(n+2)} \\ =& \frac{n^2 + 2n + 1}{(n+1)(n+2)} \\ =& \frac{n+1}{n+2}.\end{aligned}$$

Problem 7 Combinatorics (10 points)

Define a *DNA-string* to be a string over the alphabet $\{\mathrm{A}, \mathrm{T}, \mathrm{G}, \mathrm{C}\}$. Let D be the set of all DNA-strings, that is, all strings of the form $\{\mathrm{A}, \mathrm{T}, \mathrm{G}, \mathrm{C}\}^*$. For each positive integer n, let D_n be the set of all DNA-strings of length n.

a. (2 point) For a given positive integer n, how big is the set D_n?

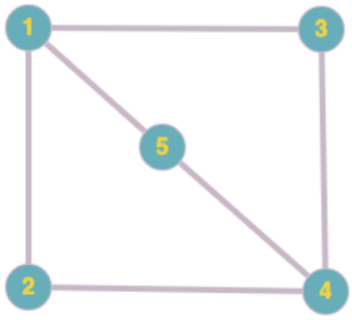

Figure 1: Graph in problem 8.

b. (3 points) How many DNA-strings in D_8 have two of each letter?

c. (5 points) How many DNA-strings in D_6 have exactly three T's?

Solution.

a. There are 4^n elements in D_n because there are n positions in each string and there are 4 possible values, A, T, G og C, for each position.

b. There are 8! permutations of $\{A, A, T, T, G, G, C, C\}$, but because we do not distinguish the strings when we swap two identical letters, we have $8!/2^4 = 2520$ different strings in total.

c. There are $\binom{6}{3}$ ways to choose where to put the three T's. For each of these, there are three positions where we can put any of the three other letters. It follows that we get $\binom{6}{3} \cdot 3^3 = 20 \cdot 27 = 540$ different strings in total.

Problem 8 Graphs and trees (10 points)

a. (4 points) Write down the adjacency matrix of the graph in the figure, where an index (i, j) in the matrix is 0 if there is no edge from vertex i to vertex j and 1 if there is an edge. Is the graph connected? Is the graph complete?

b. (6 points) Does the graph have

i) (1 point) an Euler trail?

ii) (1 point) an Euler circuit?

iii) (2 points) a Hamiltonian path?

iv) (2 points) a Hamiltonian cycle?

Solution.

a. We get the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The graph is connected since we can find a path between any pair of nodes in the graph. The graph is not complete since there is not an edge between every distinct nodes, for example, there is no edge between node 1 and 4.

b. i) Yes. We note that nodes 1 and 4 have odd degree, while the other nodes have even degree. Then there exists an Euler trail in the graph. For example: $4 \to 2 \to 1 \to 3 \to 4 \to 5 \to 1$.

ii) No. We know that a graph has an Euler circuit if and only if all nodes in the graph has even degree. Hence, it does not have an Euler circuit.

iii) Yes. But how do we find it? The path has to start or end at 5 because otherwise it must go through nodes 1 or 4 twice to reach all nodes. We can also see that it must end at 2 or 3 of the same reason. The graph has several Hamiltonian paths, for example: $2 \to 1 \to 3 \to 4 \to 5$.

iv) No. We found that a Hamiltonian path must start or end at 5 and similar for either 2 or 3, however, there are no edges between 5 and 2 or 3. Hence, the graph does not have a Hamiltonian cycle.

Problem 9 Number theory (10 points)

a. (4 points) Use Wilson's theorem, saying that $(p-1)! \equiv -1 \pmod{p}$ if and only if p is prime, to compute $10! \pmod{13}$, $11! \pmod{13}$ and $12! \pmod{13}$.

b. (6 points) Find all integer solutions x to the following system of congruences:

$$\begin{aligned} 2x + 3 &\equiv 5 \pmod{6} \\ 3x + 1 &\equiv 7 \pmod{10} \end{aligned}$$

Solution.

a. Since 13 is a prime, we can use the theorem to conclude that $12! \equiv -1 \equiv 12 \pmod{13}$. It also follows since $12! = 12 \cdot 11! \equiv (-1) \cdot 11! \equiv -1 \pmod{13}$ that $11! \equiv 1 \pmod{13}$. Finally, since $11! = 11 \cdot 10! \equiv 1 \pmod{13}$ we have that $10! \equiv 11^{-1} \equiv (-2)^{-1} \pmod{13}$. Since $(-2) \cdot (-7) = 14 \equiv 1 \pmod{13}$, then $10! \equiv 11^{-1} \equiv -7 \equiv 6 \pmod{13}$.

b. We note that $\gcd(6, 10) = 2$ so, if any solution exists, there will be a unique solution modulo $M = 6 \cdot 10/2 = 30$. We simplify the equations by subtracting free terms and get that

$$\begin{aligned} 2x &\equiv 2 \pmod{6} \\ 3x &\equiv 6 \pmod{10} \end{aligned}$$

Then, the first equation is equivalent to $2x \equiv 2 \pmod{3}$ since 6 is divisible by 2 and 3, and everything is zero modulo 2. Since $\gcd(2, 3) = 1$ we can multiply with the inverse of 2 on both sides in the first equation and since $\gcd(3, 10) = 1$ we can multiply by the inverse of 3 on both sides in the second equation. In both equations we can use division directly since both sides are divisible by the term in front of x. Then we have

$$\begin{aligned} x &\equiv 1 \pmod{3} \\ x &\equiv 2 \pmod{10} \end{aligned}$$

Now the moduli are relatively prime, so we know for sure that there exists a unique solution modulo $3 \cdot 10 = 30$. The solution is $x \equiv \sum_{i=1}^{r} a_i M_i k_i \pmod{M}$ where r is the number of congruences of the form $x \equiv a_i \pmod{m_i}$, $M = m_1 \cdot \cdots \cdot m_r$ is the product of all individual moduli, $M_i = M/m_i$, and k_i is the inverse of M_i modulo m_i. We compute $k_1 \equiv 10^{-1} \equiv 1 \pmod{3}$ and $k_2 \equiv 3^{-1} \equiv 7 \pmod{10}$. Solution: $x \equiv 10 + 2 \cdot 3 \cdot 7 = 52 \equiv 22 \pmod{30}$.

Problem 10 Finite state machines and automata (10 points)

a. (5 points) Draw the transition diagram of the finite state machine F with input $I = \{a, b, c\}$, output $O = \{0, 1, 2\}$ and states $S = \{s_0, s_1, s_2, s_3\}$ (initial state is s_0) based on the following transition table. What is the output for the input string *aabaab*?

F	η			μ		
	a	b	c	a	b	c
s_0	s_0	s_1	s_2	1	1	2
s_1	s_1	s_1	s_0	2	0	0
s_2	s_2	s_1	s_0	1	0	1
s_3	s_2	s_1	s_0	2	0	2

b. (5 points) Draw the transition diagram of an automaton with input $I = \{a, b\}$ accepting strings where every b is followed by an a.

Solution.

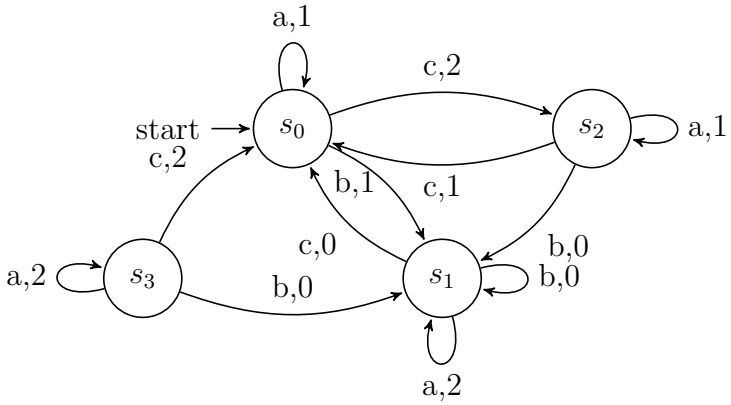

Figure 2: Solution to Problem 10.a. Output: 111220.

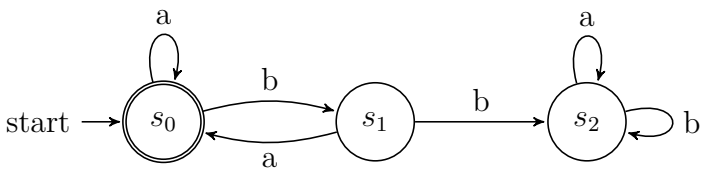

Figure 3: Solution to Problem 10.b.