**TMA4140**
**DISKRET MATEMATIKK – DISCRETE MATHEMATICS**
**NTNU, HØST/FALL2020**

Exercise Set 9 / Øving 9

The solutions must be submitted via OVSYS (to the assigned group/TA).
Løsningene må sendes inn via OVSYS (til den tildelte gruppen/TA).

Deadline for submission: **Friday, 30 October, 4:00pm**
Innleveringsfrist: **Fredag, 30. Oktober, kl. 16:00**

Textbook: K. H. Rosen, *Discrete Mathematics and Its Applications*, 8. edition

**Exercise/Oppgave**
**1.** *Consider the set* $X = \{2, 16, 128, 1024, 8192, 65536\}$*. Use the pigeonhole principle to show that if four numbers are selected from* $X$*, then two of those four numbers must have the product* 131072*. Hint: think in terms of powers of* 2*.*

*Solution.* Notice that $X = \{2, 16, 128, 1024, 8192, 65536\} = \{2^1, 2^4, 2^7, 2^{10}, 2^{13}, 2^{16}\}$ and $131072 = 2^{17}$. Notice that $X$ can be partitioned into the following three subsets: $\{2, 2^{16}\}$, $\{2^4, 2^{13}\}$ and $\{2^7, 2^{10}\}$. By the pigeonhole principle, if we choose four numbers from $X$, we will necessarily choose one of the the previous three subsets. Noticing that the product of each pair of numbers is equal to $2^{17}$, we conclude that if we choose four numbers from $X$, then two of those four numbers must have the product 131072. □

**Exercise/Oppgave**
**2.** *Use induction to show that* $\sum_{k=1}^{n}(6k-4) = n(3n-1)$*.*

*Solution.* By induction on $n$. For the base case $n = 1$, note that $\sum_{k=1}^{1}(6k-4) = 2 = 1(3 \cdot 1 - 1)$, so the base case is valid. For the induction hypothesis, assume that there exists a positive integer $m \geq 1$ such that $\sum_{k=1}^{m}(6k-4) = m(3m-1)$. We shall prove that the formula is valid for $m+1$. Indeed, by splitting

*Date*: November 2, 2020.

the sum and using the induction hypothesis, we have

$$\begin{aligned}\sum_{k=1}^{m+1}(6k-4) &= \sum_{k=1}^{m}(6k-4)+(6(m+1)-4)\\ &= m(3m-1)+6m+2\\ &= 3m^2+5m+2\\ &= (m+1)(3m+2)\\ &= (m+!)(3(m+1)-1).\end{aligned}$$

Hence the formula is valid for $m+1$. By mathematical induction, we conclude that $\sum_{k=1}^{n}(6k-4)=n(3n-1)$, for all $n\in\mathbb{N}$. □

**Exercise/Oppgave**

**3.** *Consider the sequence* $\{f_n\}_{n\geq 0}$ *of Fibonacci numbers determined for* $n>1$ *by* $f_n=f_{n-1}+f_{n-2}$, $f_0=0, f_1=1$. *Show by induction that for positive integers* $n$*:* $\sum_{i=1}^{n}(-1)^{i+1}f_{i+1}=(-1)^{n-1}f_n$.

*Solution.* By induction on $n$. For the base case $n=1$, we have $\sum_{i=1}^{1}(-1)^{i+1}f_{i+1}=f_2=1=(-1)^{1-1}f_1$, and hence the base case is true. For the induction hypothesis, assume that there exists a positive integer $k\geq$ such that $\sum_{i=1}^{k}(-1)^{i+1}f_{i+1}=(-1)^{k-1}f_k$. We will show that the formula is true for $k+1$. By splitting the sum and using the induction hypothesis, we have that

$$\begin{aligned}\sum_{i=1}^{k+1}(-1)^{i+1}f_{i+1} &= (-1)^{k-1}f_k+(-1)^{k+1+1}f_{k+1+1}\\ &= (-1)^k(f_{k+2}-f_k)\\ &= (-1)^{(k+1)-1}f_{k+1}.\end{aligned}$$

where we used that $f_{k+2}=f_k+f_{k+1}$, for all $k\geq 0$. Hence the formula is valid for $k+1$. By mathematical induction, we conclude that $\sum_{i=1}^{n}(-1)^{i+1}f_{i+1}=(-1)^{n-1}f_n$ for all $n\geq 1$. □

**Exercise/Oppgave**

**4.** *Let* $S$ *be a set and let* $P=\{A_1,\ldots,A_k\}$ *be a partition of* $S$. *We define the map* $f:S\to P$ *by* $f(s)=A_j$ *if* $s\in A_j$. *Show that* $f$ *is surjective.*

*Solution.* Recall the definition of a surjective function: $f:A\to B$ is surjective if for all $b\in B$, there is $a\in A$ such that $f(a)=b$. Consider then $f:S\to P$ as in the exercise. Take $A_j\in P$. We know, by definition of partition that $A_1,\ldots,A_k$ are non-empty, disjoint subsets of $S$ such that $\bigcup_{i=1}^{k}=S$. Since $A_j$ is non-empty, there is $s\in A$ such that $s\in A_j$. Then, by definition of $f$, we have $f(s)=A_j$. We conclude, that for any $A_j\in P$, there is $s\in S$ such that $f(s)=A_j$. Therefore $f$ is surjective. □

**Exercise/Oppgave**

**5.** *Let* $X$ *be a non-empty set and consider functions* $f,g:X\to X$. *Assume that* $f=g\circ f\circ f$ *and* $g=f\circ g\circ f$. *Show that* $f=g$.

*Solution.* Take $x \in X$. We will show that $f(x) = g(x)$. By assumption, we have

$$f(x) = g(f(f(x)) = f(g(f(f(f(x))))) = f(g \circ f^2 \circ f(x)). \tag{1}$$

We will compute $f^2(x)$. Again by assumption, we have

$$f^2(x) = f(f(x)) = f(g(f(f(x)))) = f(g \circ f^2(x)). \tag{2}$$

Plugging (2) into (1), we have

$$\begin{aligned} f(x) &= f(g \circ f^2 \circ f(x)) \\ &= f^2(f(x)) \\ &= f^2(g(f(f(x)))) \qquad \text{(using that } f = g \circ f \circ f) \\ &= f(\underbrace{f(g(f}_{=g}(f(x))))) \\ &= f(g(f(x))) \\ &= g(x). \end{aligned}$$

Therefore $f = g$. □

**Exercise/Oppgave**

**6.** *Let $A = \{1,2,3,4,5,6,7,8,9\}$. Consider the set partitions $P_1$ of $A$ with blocks $P_{11} = \{1,3,5,7,9\}$, $P_{12} = \{2,4,6,8\}$, and the set partition $P_2$ of $A$ with blocks $P_{21} = \{1,2,3,4\}$, $P_{22} = \{5,7\}$, $P_{23} = \{6,8,9\}$. Compute the set $P_3 := \{P_{1i} \cap P_{2j} \mid i = 1,2,\ j = 1,2,3\} \backslash \emptyset$. Show that $P_3$ is a partition of $A$.*

*Solution.* First compute the sets of the collection $P_3$:

$$\begin{aligned} P_{11} \cap P_{21} &= \{1,3,5,7,9\} \cap \{1,2,3,4\} = \{1,3\}, \\ P_{11} \cap P_{22} &= \{1,3,5,7,9\} \cap \{5,7\} = \{5,7\}, \\ P_{11} \cap P_{23} &= \{1,3,5,7,9\} \cap \{6,8,9\} = \{9\}, \\ P_{12} \cap P_{21} &= \{2,4,6,8\} \cap \{1,2,3,4\} = \{2,4\}, \\ P_{12} \cap P_{22} &= \{2,4,6,8\} \cap \{5,7\} = \emptyset, \\ P_{12} \cap P_{23} &= \{2,4,6,8\} \cap \{6,8,9\} = \{6,8\}. \end{aligned}$$

Hence

$$P_3 = \{\{1,3\}, \{5,7\}, \{9\}, \{2,4\}, \{6,8\}\}.$$

Now we show that $P_3$ is a partition of $A$. This is by definition since all the elements of $P_3$ are pairwise disjoint sets and their union is exactly $A$:

$$\{1,3\} \cup \{5,7\} \cup \{9\} \cup \{2,4\} \cup \{6,8\} = A.$$

□

**Exercise/Oppgave**

**7.** *Consider a surjective function $f : A \to B$. Define for $b \in B$ the set $f^{-1}(b) := \{a \in A \mid f(a) = b\} \subseteq A$. Show that $P := \{f^{-1}(b) \mid b \in B\}$ defines a partition of $A$.*

*Solution.* We have to show that the elements of $P$ are pairwise disjoint and their union is $A$. Indeed,

$$\bigcup_{b\in B} f^{-1}(b) = f^{-1}\left(\bigcup_{b\in B}\{b\}\right) = f^{-1}(B) = A,$$

by properties of the inverse image. On the other hand, consider $b, c \in B$. Then

$$f^{-1}(b) \cap f^{-1}(c) = f^{-1}(\{b\} \cap \{c\}) = \begin{cases} f^{-1}(b) & \text{if } b = c \\ \emptyset & \text{if } b \neq c \end{cases}$$

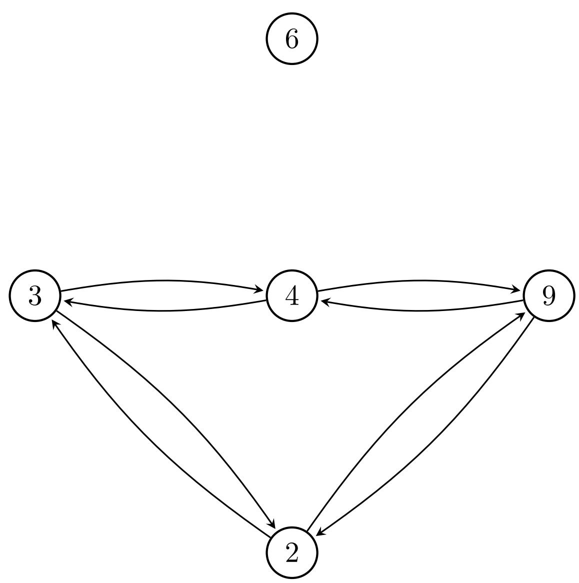
Hence the elements of P are pairwise disjoint. Finally, since $f$ is surjective, then $f^{-1}(b) \neq \emptyset$ for all $b \in B$ and hence $\emptyset \notin P$. We conclude that $P$ defines a partition of $A$. □

**Exercise/Oppgave**

**8.** *Let $A = \{2, 3, 4, 6, 9\}$. Draw the directed graph of the relation defined by*

$$R = \{(2, 3), (2, 9), (3, 2), (3, 4), (4, 3), (4, 9), (9, 2), (9, 4)\}$$

*Solution.* Recall that the directed graph $G = (V, E)$ of the relation is defined by $V = A$ and $E = R$. Then the corresponding graph of the above relation is given by




□

**Exercise/Oppgave**

**9.** Section/Sektion 9.1: *2a, 7, 42a, 42c*

*Solution.* 2a. Let $A = \{1, 2, 3, 4, 5, 6\}$ and the relation on $A$ given by $R = \{(a, b) \,|\, a \text{ divides } b\}$. Then $R$ is given by

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)\}.$$

7.

- $x \neq y$. It is not reflexive since $x = x$ for all $x \in \mathbb{Z}$. Clearly it is symmetric. It is not antisymmetric since $x \neq y$ and $y \neq x$ does not imply that $x = y$. It is not transitive since, for instance $0 \neq 1$ and $1 \neq 0$ and $0 = 0$.
- $xy \geq 1$. It is not reflexive since $0 \cdot 0 = 0 \not\geq 1$. It is symetrric since if $xy \geq 1$, then $yx = xy \geq 1$. It is not antisymmetric since $1 \cdot 2 = 2 \cdot 1 \geq 1$ and $1 \neq 2$. It is transitive. Assume that $xy \geq 1$ and $yz \geq 1$. Then we have that $y \neq 0$. Also, both inequalities imply that $xzy^2 \geq 1$, and so $xz \geq 1/y^2 > 0$. Since $x, z$ are both integers, then $xz$ is also an integer, then we get that $xz \geq 1$.
- $x = y + 1$ or $x = y - 1$. It is not reflexive since $x \neq x + 1$ and $x \neq x - 1$. It is symmetric. If $(x, y) \in R$ then $x = y + 1$ or $x = y - 1$. This is equivalent to $y = x - 1$ and $y = x + 1$. Hence $(y, x) \in R$. It is not antisymmetric since $2 = 1 + 1$ and $1 = 2 - 1$ and $1 \neq 2$. It is not transitive since $2 = 1 + 1$ and $3 = 2 + 1$ but $(1, 3) \notin R$.
- $x \equiv y \pmod 7$. It is reflexive since $x \equiv x \pmod 7$ is equivalent to $7|x - x \Rightarrow 7|0$ and this is true for all $x \in \mathbb{Z}$. It is symmetric since $x \equiv y \pmod 7$ implies that $y \equiv x \pmod 7$. It is not antisymmetric since $7 \equiv 0 \pmod 7$ and $0 \equiv 7 \pmod 7$ and $7 \neq 0$. It is transitive: if $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$, then $7|(x - y)$ and $7|(y - z)$. This implies that $7|(x - y + y - z) \Rightarrow 7|(x - z) \Rightarrow x \equiv z \pmod 7$.
- $x$ is a multiple of $y$. It is reflexive since $x$ is multiple of $x$ for all $x \in \mathbb{Z}$ since $x = 1 \cdot x$. It is not symmetric: 2 is clearly multiple of 1 but 1 is not multiple of 2. It is not antisymmetric, since 1 is multiple of $-1$, $-1$ is multiple of 1, but $1 \neq -1$. It is transitive: if $x$ is multiple of $y$ and $y$ is multiple of $z$ then there are $s, t$ integers such that $x = yt$ and $y = sz$. Then $x = yt = z(st)$. This implies that $x$ is multiple of $z$.
- $x$ and $y$ are both negative or both nonnnegative. It is reflexive $x$ and $x$ have the same sign for all $x \in \mathbb{Z}$. It is symmetric: if $x, y$ are both negative or both nonnegative then $y, x$ are both negative or both nonnegative. It is not antisymmetric since $1, 2$ satisfy that $(1, 2), (2, 1) \in R$ but $1 \neq 2$. It is transitive: if $x, y$ have the same sign and $y, z$ have the same sign, then $x, y, z$ have the same sign. This implies that $x, z$ have the same sign, i.e. $(x, z) \in R$.
- $x = y^2$. It is not reflexive since $2 \neq 4 = 2^2$. It is not symmetric since $4 = 2^2$ but $2 \neq 4^2$. It is antisymmetric $x = y^2$ and $y = x^2$ imply that $x = x^4$ and this implies that $x = 1$ or $x = 0$. Hence $y = 1$ or $y = 0$, respectively, so $x = y$. It is not transitive: $4 = 2^2$ and $16 = 4^2$ and $16 \neq 2^2$.
- $x \geq y^2$. It is not reflexive since $2 \not\geq 2^2$. It is not symmetric since $9 \geq 3^2$ but $3 \not\geq 9^2$. It is antisymmetric: if $x \geq y^2$ and $y \geq x^2$ then $x \geq x^4$. This implies that $x \geq 0$. If $x = 0$ then $y = 0$ and so $x = y$. If $x > 0$ then $1 \geq x^3$ and this implies that $x = 1$, so $y = 1$. Hence $x = y$. It is transitive: if $x \geq y^2$ and $y \geq z^2$, then $x \geq z^4 \geq z^2$.

42a. Observe that $a$ divides $b$ if and only if $b$ is multiple of $a$. Then

$$R_1 \cup R_2 = \{(a,b) \in \mathbb{N}^2 \mid \exists c \in \mathbb{N}: \ a = bc \vee b = ac\}.$$

42c. $R_1$ is the relation $a$ divides $b$. Notice that $a$ divides $b$ and $a$ is multiple of $b$ if and only if $a = bs$ for some $s$ and $b = at$ for some $t$. Then we have $a = bs = ats$. This implies that $st = 1$, so $s = t = 1$ and $a = b$. Then

$$R_1 - R_2 = \{(a,b) \in \mathbb{N}^2 \mid \exists\, c \in \mathbb{N}, c \neq 1 : a = bc\}.$$

□

**Exercise/Oppgave**
**10.** Section/Sektion 9.3: *10, 14, 22*

*Solution.* 10.

- $\{(a,b) \mid a \geq b\}$. The nonzero entries correspond to the entries in the diagonal and below of it. This number is $1 + 2 + \cdots + 1000 = 500500$.
- $\{(a,b) \mid a = b \pm 1\}$. The nonzero entries are the subdiagonal below and above of the main diagonal: $(i, i+1)$ and $(i-1, i)$. The number of nonzero entries is $999 + 999 = 1998$,
- $\{(a,b) \mid a + b = 1000\}$. This corresponds to the entries $(a, 1000 - a)$, for $a \in \{1, \ldots, 999\}$. Hence the number is 999.
- $\{(a,b) \mid a + b \leq 1001\}$. This corresponds to the entries $(a, 1001 - b)$ for $a \in \{1, \ldots, 1000\}$ and $b \in \{1, \ldots, a\}$. Then the number of nonzero entries is $1 + 2 + \cdots + 1000 = 500500$.
- $\{(a,b) \mid a \neq 0\}$. Since $A = \{1, \ldots, 1000\}$, then if $a \in A$ we have that $a \neq 0$. Then all the entries of the matrix are nonzero and this number is $1000^2 = 1000000$.

14.

- $R_1 \cup R_2$. We know that $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$. Hence

$$M_{R_1 \cup R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

- $R_1 \cap R_2$. We know that $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$. Hence

$$M_{R_1 \cap R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

- $R_2 \circ R_1$. We know that $M_{R_2 \circ R_1} = M_{R_1} \odot M_{R_2}$. Hence

$$M_{R_2 \circ R_1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

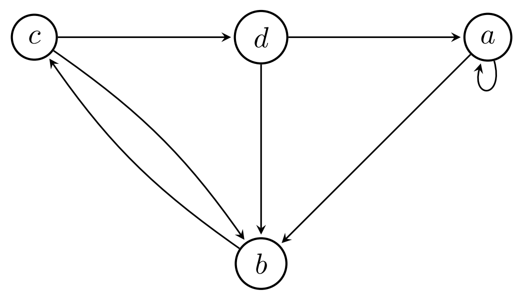
- $R_1 \circ R_1$. We know that $M_{R_1 \circ R_1} = M_{R_1} \odot M_{R_1}$. Hence

$$M_{R_1 \circ R_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- $R_1 \oplus R_2$. We know that $M_{R_1 \oplus R_2} = M_{R_1 \cup R_2} - M_{R_1 \cap R_2}$. Hence

$$M_{R_1 \oplus R_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

22.




□