TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

Solutions Exercise Set 8

Exercise/Oppgave

1. *i) How many ways are there to select 12 cars if Fords, BMWs, and Fiats are available?*
ii) How many if at least 1 car of each type must be selected?
iii) If at least 3 Fiats have to be selected?
iv) If at least 1 car of each type and at least 4 BMWs have to be selected?

Solution. (1) This number is given by the number of combinations with repetition of taking 12 objects from 3 different type of objects, i.e.

$$C(12+3-1,12) = \binom{14}{12} = 91.$$

(2) A way of selecting 12 cars with the given restrictions corresponds to select one Ford, one BMW, one Fiat, together with a choice of 9 additional cars of any brand. This can be done in

$$C(9+3-1,9) = \binom{11}{9} = 55.$$

ways.

(3) In a similar way that the previous item, a way of selecting 12 cars taking at least 3 Fiats corresponds to take 3 Fiats and a choice of 9 additional cars of any brand. This can be done in

$$C(9+3-1,9) = \binom{11}{9} = 55.$$

ways.

(4) In a similar way, we have to select 1 Ford, 4 BMWs, 1 Fiat, and a choice of 6 additional cars of any brand. This can be done in

$$C(6+3-1,6) = \binom{8}{6} = 28$$

ways.

□

Exercise/Oppgave

2. *How many distinct non-negative integer solutions has the equation*

$$x_1 + x_2 + x_3 + x_4 = 12.$$

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Solution. Since $x_1, \dots, x_4$ are non-negative integers, we can interpret x_i as the number of objects of type i, for $i = 1, 2, 3, 4$. Then the equation is equivalent to count the number of ways that we can take 12 objects of four different type of objects, i.e. combinations with repetition. This number is given by

$$C(12 + 4 - 1, 12) = \binom{15}{12} = 455.$$

□

Exercise/Oppgave
3. *Show that $f \in \Omega(g)$ for $f(n) = 3n^2 - 5$ and $g(n) = n(n+2)$.*

Proof. Recall the definition: $f \in \Omega(g)$ if there exist $k, C > 0$ constants such that $|f(n)| \geq C|g(x)|$ whenever $x > k$. Take $k = 2$. Note that if $n > k$ then:

- $2n \leq n^2$: just multiply the inequality $2 < n$ by $n > 0$,
- $n^2 - 5 \geq 0$: this is equivalent to $n^2 \geq 5 > 4$ and this is true since $n > 2$.

Hence

$$\begin{aligned} |g(n)| &= n(n+2) \\ &= n^2 + 2n \\ &\leq n^2 + n^2 \\ &= 2n^2 \\ &\leq 2n^2 + (n^2 - 5) \\ &= 3n^2 - 5 \\ &= |f(n)|, \end{aligned}$$

since $f(n) \geq 0$ for $n > k$. Taking $C = 1$, we have that $|f(n)| \geq C|g(n)|$ for $n > k$, and hence $f \in \Omega(g)$. □

Exercise/Oppgave
4. Section/Sektion 5.1: *4, 6, 9, 10, 14*

Solution. 4. We will show the statement by using mathematical induction.

(1) The statement $P(1)$ is given by

$$1^3 = (1(1+1)/2)^2.$$

(2) Note that $1^3 = 1$ and $(1(1+1)/2)^2 = (1 \cdot 2/2)^2 = 1^2 = 1$, and hence $P(1)$ is true.
(3) For the induction hypothesis, assume that $P(k)$ is true for a positive integer k, i.e., there exists $k \geq 1$ such that

$$1^3 + 2^3 + \cdots + k^3 = (k(k+1)/2)^2.$$

(4) For proving the inductive step, we need to prove that $P(k+1)$ is true, i.e., we will prove that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = ((k+1)((k+1)+1)/2)^2 = ((k+1)(k+2)/2)^2.$$

(5) Now we will prove it. Notice that

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \left(1^3+2^3+\cdots+k^3\right)+(k+1)^3\\
&= (k(k+1)/2)^2+(k+1)^3\\
&= (k+1)^2\left(k^2/4+k+1\right)\\
&= (k+1)^2(k^2+4k+4)/4\\
&= (k+1)^2(k+2)^2/4\\
&= ((k+1)(k+2)/2)^2.
\end{aligned}$$

Note that we used the induction hypothesis on the second equality. We conclude that $P(k+1)$ is true.

(6) For the principle of mathematical induction, we conclude that $P(n)$ is true for every n positive integer. By a), $P(1)$ is true. By e) we have that, when $P(k)$ is true, then $P(k+1)$ is true. Since $P(1)$ is true, then $P(2)$ is true. Since $P(2)$ is true, then $P(3)$ is true. Since the positive integers is a countable set, the process reaches every $n \geq 1$, making $P(n)$ true for every $n \geq 1$.

6. We will prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n! = (n+1)!-1$ for every positive integer n. For the base step, notice that $1\cdot 1! = 1 = (1+1)!-1$ and so the base case is true. For the induction hypothesis, assume that there exists a positive integer k such that $1\cdot 1!+2\cdot 2!+\cdots+k\cdot k! = (k+1)!-1$. We will prove that the statement is true, i.e., $1\cdot 1!+2\cdot 2!+\cdots+(k+1)\cdot(k+1)! = ((k+1)+1)!-1$. Indeed, by splitting the sum and using the induction hypothesis, we have

$$\begin{aligned}
1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)! &= (k+1)!-1+(k+1)\cdot(k+1)!\\
&= (k+1)!(1+k+1)-1\\
&= (k+1)!(k+2)-1\\
&= (k+2)!-1\\
&= ((k+1)+1)!-1.
\end{aligned}$$

Hence the induction step is complete. By mathematical induction, we conclude that the initial statement is true for every positive integer n.

9. a,b) We will find a formula for the sum of the first n even positive integers. For the first cases, we have the following computations:

- $2 = 2 = 1\cdot 2$,
- $2+4 = 6 = 2\cdot 3$,
- $2+4+6 = 12 = 3\cdot 4$,
- $2+4+6+8 = 20 = 4\cdot 5$,
- $2+4+6+8+10 = 30 = 5\cdot 6$.

Hence, we conjecture that $2+4+6+\cdots+2n = n(n+1)$. We will prove it by mathematical induction. For the base case, note that $2 = 1(1+1)$ and this is true. Now, assume that there exists a positive integer k such that $2+4+\cdots+2k = k(k+1)$. We will prove that the formula is true for $k+1$, i.e., we shall prove that $2+4+\cdots+2k+2(k+1) = (k+1)((k+1)+1)$. Indeed, by splitting the sum and using the induction hypothesis, we have

$$\begin{aligned} 2+4+\cdots+2k+2(k+1) &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)((k+1)+1). \end{aligned}$$

Hence the formula is true for $k+1$. By mathematical induction, we conclude that $2+4+6+\cdots+2n = n(n+1)$ for every $n \in \mathbb{N}$.

10. a,b) Notice that, for small values of n, we have

- $\frac{1}{1\cdot 2} = \frac{1}{2}$,
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}$,
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}$,
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$,
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}$.

We conjecture that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$, for every positive integer n. We will prove it by induction. The base case follows from the above computations. For the induction hypothesis, assume that there exists $k \geq 1$ such that the formula holds. We will prove that the formula is true for $k+1$. By splitting the sum and using the induction hypothesis we have

$$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{1}{k+1}\left(k + \frac{1}{k+2}\right) \\ &= \frac{1}{k+1}\,\frac{k(k+2)+1}{k+2} \\ &= \frac{1}{k+1}\,\frac{(k+1)^2}{k+2} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1}. \end{aligned}$$

Hence the formula is true for $k+1$. By induction, we conclude that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for every $n \in \mathbb{N}$.

14. We will prove that for every positive integer n,

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2.$$

By induction. Base case: for $n=1$, we have that $2 = \sum_{i=1}^{1} 2^1 \cdot 1 = (1-1)2^{1+1} + 2 = 0 + 2 = 2$. So, the base case is correct.

Induction hypothesis: assume that the statement is true for some positive integer $k \geq 1$, i.e. $\sum_{i=1}^{k} 2^i i = (k-1)2^{k+1} + 2$.

Induction step: we will prove that the statement is true for $k+1$. By splitting the sum and using the induction hypothesis, we have that

$$\begin{aligned}\sum_{i=1}^{k+1} 2^i i &= \sum_{i=1}^{k} 2^i i + 2^{k+1}(k+1) \\ &= 2^{k+1}(k-1) + 2 + 2^{k+1}(k+1) \qquad \text{(by induction hypothesis)} \\ &= 2^{k+1}((k-1)+(k+1)) + 2 \\ &= 2^{k+1}(2k) + 2 \\ &= 2^{k+2}k + 2 \\ &= 2^{(k+1)+1}((k+1)-1) + 2.\end{aligned}$$

Hence, the statement is true for $k+1$. We conclude by mathematical induction that for any $n \geq 1$, $\sum_{i=1}^{n} 2^i i = 2^{n+1}(n-1) + 2$. □

Exercise/Oppgave

5. *Use induction to show that for all natural numbers m*

$$\sum_{i=1}^{m}(4i^3 + 24i^2 + 32i) = m(m+5)(m+4)(m+1).$$

Solution. We proceed by mathematical induction on m. For the base case $m = 1$, note that

$$\sum_{i=1}^{1}(4i^3 + 24i^2 + 32i) = 4 + 24 + 32 = 60 = 1(1+1)(1+4)(1+5),$$

and then the base case holds. For the induction hypothesis, assume that the formula holds for a natural number k. We will show that the formula holds for $k+1$. Indeed, by splitting the sum and using the induction hypothesis, we have

$$\begin{aligned}\sum_{i=1}^{k+1}(4i^3 + 24i^2 + 32i) &= k(k+1)(k+4)(k+5) + 4(k+1)^3 + 24(k+1)^2 + 32(k+1) \\ &= k(k+1)(k+4)(k+5) + 4(k+1)((k+1)^2 + 6(k+1) + 8) \\ &= (k+1)[k(k+4)(k+5) + 4(k^2 + 8k + 15)] \\ &= (k+1)[k(k+4)(k+5) + 4(k+3)(k+5)] \\ &= (k+1)(k+5)[k(k+4) + 4(k+3)] \\ &= (k+1)(k+5)(k^2 + 8k + 12) \\ &= (k+1)(k+5)(k+2)(k+6) \\ &= (k+1)(k+2)(k+5)(k+6) \\ &= (k+1)((k+1)+1)((k+1)+4)((k+1)+5).\end{aligned}$$

Hence the formula holds for $k+1$. By mathematical induction, we conclude that the formula holds for all natural numbers $n \in \mathbb{N}$. □

Exercise/Oppgave

6. *Determine the smallest non-negative integer such that* $\frac{1}{n+1}\binom{2n}{n} > n+2$ *and then show by induction that the formula holds in general.*

Proof. By induction on n, we will prove that the statement holds for $n \geq$. It is easy to see that

- $\frac{1}{2}\binom{2}{1} = 1 \not> 3$,
- $\frac{1}{3}\binom{4}{2} = 2 \not> 4$,
- $\frac{1}{4}\binom{6}{3} = 5 \not> 5$.

For the base case $n = 4$ we have that $\frac{1}{4+1}\binom{8}{4} = 14 > 6 = 4+2$. For the induction hypothesis, assume that the inequality holds for a positive integer $k \geq 1$. We will prove that the inequality still holds for $k+1$. Observe that

$$\begin{aligned}
\frac{k+1}{\binom{2(k+1)}{k+1}} &= \frac{1}{k+2}\frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} \\
&= \frac{(2k+2)(2k+1)}{(k+2)(k+1)} \cdot \frac{1}{k+1}\frac{(2k)!}{k!k!} \\
&= \frac{(2k+2)(2k+1)}{(k+2)(k+1)} \cdot \frac{1}{k+1}\binom{2k}{k} \\
&= \frac{(2k+2)(2k+1)}{(k+2)(k+1)}(k+2) \\
&= 2(2k+1) \\
&= 4k+2.
\end{aligned}$$

Finally, notice that the inequality $k+1 < 4k$ is equivalent to $1 < 3k$, and since $k \geq 1$, then $3k \geq 3 > 1$ and so $k+1 < 4k$ is true. Hence $4k+2 > (k+1)+1$ and we completed the induction step. By mathematical induction, we conclude that the smallest non-negative integer n_0 such that $\frac{1}{n+1}\binom{2n}{n} > n+2$ for all $n \geq n_0$ is $n_0 = 4$. □

Exercise/Oppgave

7. Section/Sektion 5.2: *4, 7, 14, 19, 23*

Solution. 4.

(1) We observe that
   - $P(18)$ is true since $18 = 2 \cdot 7 + 4$,
   - $P(19)$ is true since $19 = 7 + 3 \cdot 4$,
   - $P(20)$ is true since $20 = 5 \cdot 4$,
   - $P(21)$ is true since $21 = 3 \cdot 7$.

(2) The inductive hypothesis of a proof by strong induction is then following: there exists a positive integer $k \geq 18$ such that a postage of m cents can be formed using just 4-cent stamps and 7-cent stamps, for all $18 \leq m \leq k$.

(3) Induction step: a postage of $k+1$ cents can be formed using just 4-cent stamps and 7-cent stamps.

(4) We now prove that the induction step is true for $k \geq 21$. Observe that $k+1 > k \geq k-3 \geq 18$. Hence, by the induction hypothesis, there exists non-negative integers a, b such that $k-3 =$

$4a+7b$, i.e., a postage of $k-3$ cents can be formed using a 4-cent stamps and b 7-cent stamps. Noticing that $k+1 = 4+k-3 = 4+4a+7b = 4(a+1)+7b$, we have that a postage of $k+1$ cents can be formed using $a+1$ 4-cent stamps and b 7-cent stamps. This completes the induction step.

(5) The above development shows that $P(n)$ is true for all integers $n \geq 18$ by strong induction. Part a) together the fact the fact that $P(18), P(19), \cdots, P(k)$ implies that $P(k+1)$ is true, tell us that $P(18), P(19), P(20), P(21)$ implies that $P(22), P(23), P(24), P(25)$ are true. The above implies that $P(26), P(27), P(28), P(29)$ are true. Continuing this process, we conclude that $P(n)$ is true for all positive integers $n \geq 18$.

7. Let $P(n)$ be the statement that n dollars can be formed by using just two-dollar bills and five-dollar bills. We will prove that $P(n)$ is true for every positive integer $n \geq 4$ by strong induction. For the base case, notice that $P(4) = 2 \cdot 2$, i.e. we can form 4 dollars by using 2 two-dollar bills. We also have that $P(5)$ is true since we can form 5 dollars by using 1 five-dollar bills. Now, assume that there is $k \geq 5$ such that $P(m)$ is true for all $4 \leq m \leq k$, i.e. we can form m dollars by using only two dollar bills and five-dollars bills, for $4 \leq m \leq k$. We will prove that we can do the same for $k+1$. Indeed, notice that, since $k \geq 5$, then $k > k-1 \geq 4$. Then, we can use the induction hypothesis in order to find two non-negative integers a, b such that $k-1 = 2a+5b$, i.e., we can form $k-1$ dollars by using a two-dollar bills and b five-dollar bills. Observe that

$$k+1 = (k-1)+2 = 2a+5b+2 = 2(a+1)+5b.$$

Above development implies that we can form $k+1$ dollars by using $a+1$ two-dollar bills and b five-dollar bills and hence $P(k+1)$ is true. By strong induction, we conclude that $P(n)$ is true for all integer $n \geq 4$. Finally, we clearly have that for $n = 1, 3$ we cannot find such combination, but we can find it for $n = 2$.

14. We will prove the statement by strong induction. For the base case $n = 1$, notice that we do not have to do any splitting, then the sum of all the products is an empty sum and it is equal to $0 = 1(1-1)/2$. We can also check the case $n = 2$, we can only do one split by taking two small piles of one stone each one. The product in this step is equal to $1 \cdot 1 = 1$ and so the corresponding sum is $1 = 2(2-1)/2$. We also notice that no matter how we did the splitting, the sum is constant. This proves the base case. Now, assume that there exists a positive integer $k \geq 2$ such that the statement is true for all $1 \leq m \leq k-1$. We will prove the result for k. Let $S(m)$ be the sum of the products computed at each step where we began with m stones, for $1 \leq m \leq k-1$. By strong induction hypothesis, we have that $S(m) = m(m-1)/2$ for $1 \leq m \leq k-1$. Let $S(k, m)$ be the sum of the products computed at each step where we began with k stones and we did the first split into one pile of m stones and one pile of $k-m$ stones. We would like to prove that $S(k, m) = k(k-1)/2$, for all $1 \leq m \leq k-1$. Indeed, assume that we begin with k stones and the first split was done by taking m stones. The product of this first step is then equal to $m(k-m)$. Now, we will do the same for the two remaining piles of m and $k-m$ stones. We can think that now we begin two new processes but, since $1 \leq m, k-m \leq k-1$, the corresponding sum of the products in the two sub-piles is given

by $S(m) = m(m-1)/2$ and $S(k-m) = (k-m)(k-m-1)/2$, respectively. Hence

$$\begin{aligned}
S(k,m) &= m(k-m) + S(m) + S(k-m)\\
&= m(k-m) + m(m-1)/2 + (k-m)(k-m-1)/2\\
&= \frac{m}{2}(2k-2m+m-1) + (k-m)(k-m-1)/2\\
&= \frac{km}{2} + \frac{m(k-m-1)}{2} + \frac{(k-m)(k-m-1)}{2}\\
&= \frac{km}{2} + \frac{(k-m-1)(m+k-m)}{2}\\
&= \frac{km+(k-m-1)k}{2}\\
&= \frac{km+k^2-km-k}{2}\\
&= \frac{k^2-k}{2}\\
&= \frac{k(k-1)}{2}.
\end{aligned}$$

Observe that $S(k,m)$ does not depend of m. We have proved that the statement is true for k as we wanted. By strong mathematical induction, we conclude that no matter how we split the piles, the sum of the products computed at each step equals $n(n-1)/2$, for $n \geq 1$.

19. We will prove the statement by strong induction on n, the number of sides of a simple polygon P. For the base case of triangles, $n = 3$, we will prove the formula for the following particular cases:

- The formula is true for rectangles whose sides are parallel to the coordinates axes, since, if we have a rectangle R whose vertices have integer coordinates $(x,y), (x,y+b), (x+a,y)$ and $(x+a,y+b)$, where $x,y \in \mathbb{Z}$ and $a,b \in \mathbb{N}$, the area is clearly given by ab. On the other hand, it is clear that $B(R) = 2a+2b$ and $I(R) = (a-1)(b-1)$. Hence

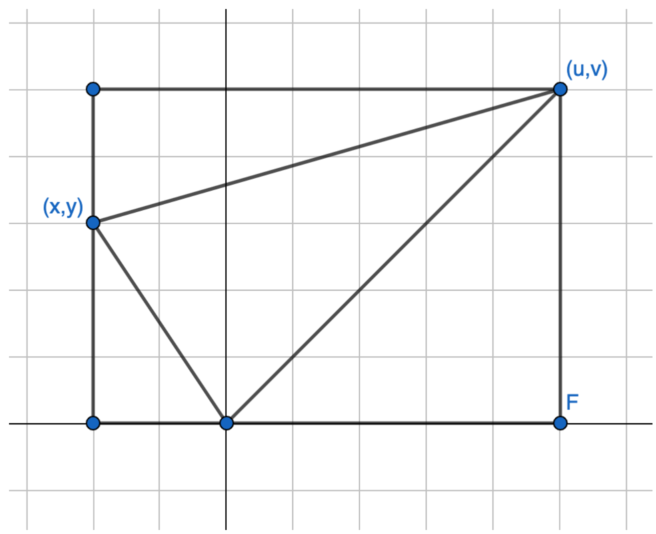
$$I(R) + B(R)/2 - 1 = ab - a - b + 1 + (2a+2b)/2 - 1 = ab,$$

  and then the theorem is true for rectangles.
- The formula is true for right triangles whose sides that form the right angle are parallel to the coordinate axes. By translating and rotating, assume that the coordinates of the right triangle T are $(0,0), (a,0)$ and $(0,b)$, for $a,b \in \mathbb{N}$. The area of this triangle is $ab/2$. Let s be the number of lattice points that lie in the segment from $(a,0)$ to $(0,b)$ without consider the extreme points. Then we have that $B(T) = a+b+s+1$. By considering the rectangle defined by the vertices of the triangle and the coordinate (a,b), we have that $I(T)$ is equal to the half of interior points points of the rectangle minus the points in the diagonal from

$(a, 0)$ to $(0, b)$, since such points are not in the interior of T. Then we conclude

$$\begin{aligned} I(T) + B(T)/2 - 1 &= \frac{(a-1)(b-1) - s}{2} + \frac{a+b+s+1}{2} - 1 \\ &= \frac{ab - a - b + 1 - s + a + b + s + 1 - 2}{2} \\ &= \frac{ab}{2}. \end{aligned}$$

- We can prove now the base case. By translating, we assume that the coordinates of the triangle T are $(0, 0)$, (x, y) and (u, v), where $x, y, u, v \in \mathbb{Z}$. Observe that we can construct a rectangle whose sides are parallel to the coordinate axes as follows:



Observe that this procedure gives us a rectangle R divided into three (depending of T) or four triangles: three right triangles T_1, T_2 and T_3 and the original one T. Assume that we have three right triangles. The argument with only having two right triangles is similar. Now, if a, b and c are the number of lattice points on each side of the triangle T without considering the vertices, it is easy to see that

$$\begin{aligned} I(R) &= I(T) + I(T_1) + I(T_2) + I(T_3) + a + b + c, \\ B(R) &= B(T_1) + B(T_2) + B(T_3) - a - b - c - 3 \\ &= B(T_1) + B(T_2) + B(T_3) + B(T) - 2a - 2b - 2c - 6, \end{aligned}$$

where we used that $B(T) = a + b + c + 3$. Hence

$$\begin{aligned} I(T) + B(T)/2 - 1 &= I(R) - I(T_1) - I(T_2) - I(T_3) - a - b - c \\ &\quad + (B(R) - B(T_1) - B(T_2) - B(T_3) + 2a + 2b + 2c + 6)/2 - 1 \\ &= (I(R) + B(R)/2 - 1) - (I(T_1) + B(T_1)/2 - 1) \\ &\quad - (I(T_2) + B(T_2)/2 - 1) - (I(T_3) + B(T_3)/2 - 1) \\ &= \text{Area}(R) - \text{Area}(T_1) - \text{Area}(T_2) - \text{Area}(T_3) \\ &= \text{Area}(T). \end{aligned}$$

and then the base case is proved.

Now, for the induction hypothesis, assume that there exists $k \geq 4$ such that the Pick's formula holds for simple polygons of m sides, for $1 \leq m \leq k-1$. We will prove that the formula also holds for a polygon P of k sides. By Lemma 1, since $k \geq 4$ there exists an interior diagonal. This diagonal divides P into two smaller polygons P_1 and P_2, where P_1 has m sides for some $3 \leq m \leq k-1$. Let s be the number of lattice points that the above interior diagonal has on it without consider the extreme points. Since also $3 \leq k-m \leq k-1$, by the induction hypothesis we have

$$\begin{aligned} I(P) &= I(P_1) + I(P_2) + s, \\ B(P) &= B(P_1) + B(P_2) - 2s - 2. \end{aligned}$$

Hence

$$\begin{aligned} I(P) + B(P)/2 - 1 &= I(P_1) + I(P_2) + s + B(P_1)/2 + B(P_2)/2 - s - 1 - 1 \\ &= I(P_1) + B(P_1)/2 - 1 + I(P_2) + B(P_2) - 1 \\ &= \text{Area}(P_1) + \text{Area}(P_2) \\ &= \text{Area}(P). \end{aligned}$$

□

Hence the formula is true for simple polygons with k sides, and by strong induction, the theorem is proved.

23. Let $E(n)$ be the statement that in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon. For the part a), an strong induction proof runs into difficulties since, when we consider the internal diagonal of P given by Lemma 1 and the resulting polygons P_1 and P_2, it is possible that there is an only triangle with two sides bordering the exterior of P_i and one of these two sides is precisely the internal diagonal, for $i = 1, 2$. This does not allow to conclude that, for the triangulation of P given by such triangulations of P_1 and P_2, there is a triangle with two sides bordering the exterior of P. However, if we assume the stronger statement $T(n)$, even in the case previously described, there is at least one triangle in P_1 whose two sides bordering the exterior are different from the internal diagonal. The same happens of P_2. This give us two triangles in the triangulation that have two sides bordering the exterior polygon of the polygon, completing the inductive step. The base case $n = 4$ is clear, proving that $T(n)$ is true for every $n \geq 4$.

Exercise/Oppgave

8. Section/Sektion 5.3 *12, 13, 14, 15, 18*

Solution. 12. By induction. For the base case $n = 1$, observe that $f_1^2 = 1^2 = 1$ and $f_n f_{n+1} = 1 \cdot 1 = 1$. Now, assume that the formula is true for a positive integer $k \geq 1$, i.e. $f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$. We will prove that the formula also holds for $k + 1$. By splitting the sum and using the induction hypothesis we have

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2} \\ &= f_{k+1} f_{(k+1)+1}, \end{aligned}$$

where we used the definition of the Fibonacci numbers $f_k + f_{k+1} = f_{k+2}$. We proved that the formula holds for $k+1$. By mathematical induction, we conclude that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for every positive integer n.

13. By induction. For the base case $n = 1$, observe that $f_1 = 1$ and $f_{2\cdot 1} = f_2 = 1$. Now, assume that the formula is true for a positive integer $k \geq 1$, i.e. $f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$. We will prove that the formula also holds for $k+1$. By splitting the sum and using the induction hypothesis we have

$$\begin{aligned} f_1 + f_3 + \cdots + f_{2k-1} + f_{2(k+1)-1} &= f_{2k} + f_{2k+1} \\ &= f_{2k+2} \\ &= f_{2(k+1)}, \end{aligned}$$

where we used the definition of the Fibonacci numbers $f_{2k} + f_{2k+1} = f_{2k+2}$. We proved that the formula holds for $k+1$. By mathematical induction, we conclude that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for every positive integer n.

14. By induction on n. For the base case $n = 1$, notice that $f_0 f_2 - f_1^2 = 0 \cdot 1 - 1^2 = -1 = (-1)^1$. Now, for the induction hypothesis, assume that there exists an integer $k \geq 1$ such that $f_{k+1} f_{k-1} - f_k^2 = (-1)^k$. We will prove that the statement is true for $k+1$. By the definition of the Fibonacci numbers, note that

$$\begin{aligned} f_{k+2} f_k &= (f_{k+1} + f_k)(f_{k+1} - f_{k-1}) \\ &= f_{k+1}^2 + f_k f_{k+1} - f_{k+1} f_{k-1} - f_k f_{k-1} \\ &= f_{k+1}^2 + f_k(f_k + f_{k-1}) - f_{k+1} f_{k-1} - f_k f_{k-1} \\ &= f_{k+1}^2 + f_k^2 - f_{k+1} f_{k-1} \end{aligned}$$

Hence

$$\begin{aligned} f_{(K+1)+1} f_{(k+1)-1} - f_{k+1}^2 &= f_{k+2} f_k - f_{k+1}^2 \\ &= f_{k+1}^2 + f_k^2 - f_{k+1} f_{k-1} - f_{k+1}^2 \\ &= f_k^2 - f_{k+1} f_{k-1} \\ &= -(-1)^k \qquad \text{(by induction hypothesis)} \\ &= (-1)^{k+1}. \end{aligned}$$

So, the formula is true for $k+1$. By mathematical induction, we conclude that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ for every positive integer n.

15. By induction on n. For the base case $n = 1$, observe that $f_0 f_1 + f_1 f_2 = 0\cdot 1 + 1\cdot 1 = 1 = f_2^2$. Now, assume that the formula is true for a positive integer $k \geq 1$, i.e. $f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} = f_{2k}^2$. We will show that the formula is also true for $k+1$. By splitting the sum and using the induction hypothesis, we have

$$\begin{aligned}
f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} &= f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= (f_{2k+2} - f_{2k+1})^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= f_{2k+2}^2 + f_{2k+1}^2 - 2 f_{2k+1} f_{2k+2} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= f_{2k+2}^2 + f_{2k+1}^2 - f_{2k+1} f_{2k+2} + f_{2k} f_{2k+1} \\
&= f_{2k+2}^2 + f_{2k+1}^2 - f_{2k+1}(f_{2k} + f_{2k+1}) + f_{2k} f_{2k+1} \\
&= f_{2k+2}^2 + f_{2k+1}^2 - f_{2k+1} f_{2k} - f_{2k+1}^2 + f_{2k} f_{2k+1} \\
&= f_{2(k+1)}^2.
\end{aligned}$$

Hence the formula is true for $k+1$. By mathematical induction, we conclude that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ for ever positive integer n.

18. By induction on n. For the base case $n = 1$, observe that

$$A^1 = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = A.$$

Now, assume that there exists a positive integer k such that

$$A^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}.$$

We will show that the statement also holds for $k+1$. Indeed, notice that

$$\begin{aligned}
A^{k+1} &= A A^k \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} \\ f_{k+1} + 0 & f_k + 0 \end{pmatrix} \\
&= \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix},
\end{aligned}$$

where we used that $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$. So, the formula holds for $k+1$. By mathematical induction, we conclude that the statement is true for all $n \geq 1$. $\square$