TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

Solutions Exercise Set 7

Exercise/Oppgave

1. *Let A, B, C be sets. Consider relations $X \subseteq A \times B$ and $Y \subseteq B \times C$. We define the composition of relations X and Y as the relation $Y \circ X \subseteq A \times C$ consisting of ordered pairs (a, c), for which there exists an element $b \in B$ such that $(a, b) \in X$ and $(b, c) \in Y$.*

We define for a relation $W \subseteq A \times B$ the inverse relation $W^{-1} := \{(b, a) \mid (a, b) \in W\} \subseteq B \times A$.

Consider the relations $R, S \subset \mathbb{N} \times \mathbb{N}$ defined by:

$$R = \{(0, 2), (0, 5), (0, 9), (1, 9), (1, 12), (1, 15), (2, 2)\}$$

$$S = \{(2, 0), (2, 6), (5, 6), (9, 8), (12, 1), (12, 7), (15, 4)\}.$$

Determine R^{-1}, S^{-1} and $(S \circ R)^{-1}$ and deduce a connection between the three relations?

Solution. According to the definition of composition of relation, the list of elements in $S \circ R$ is the following:

$$\begin{aligned}
(0, 2) \in R, (2, 0) \in S &\quad\Rightarrow\quad (0, 0) \in S \circ R,\\
(0, 2) \in R, (2, 6) \in S &\quad\Rightarrow\quad (0, 6) \in S \circ R,\\
(0, 9) \in R, (9, 8) \in S &\quad\Rightarrow\quad (0, 8) \in S \circ R,\\
(1, 9) \in R, (9, 8) \in S &\quad\Rightarrow\quad (1, 8) \in S \circ R,\\
(1, 12) \in R, (12, 1) \in S &\quad\Rightarrow\quad (1, 1) \in S \circ R,\\
(1, 12) \in R, (12, 7) \in S &\quad\Rightarrow\quad (1, 7) \in S \circ R,\\
(1, 15) \in R, (15, 4) \in S &\quad\Rightarrow\quad (1, 4) \in S \circ R,\\
(2, 2) \in R, (2, 0) \in S &\quad\Rightarrow\quad (2, 0) \in S \circ R,\\
(2, 2) \in R, (2, 6) \in S &\quad\Rightarrow\quad (2, 6) \in S \circ R.
\end{aligned}$$

$$S \circ R = \{(0, 0), (0, 6), (0, 8), (1, 8), (1, 1), (1, 7), (1, 4), (2, 0), (2, 6)\}.$$

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By definition, we have that $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$. Hence the inverses of the three above relations are

$$\begin{aligned} R^{-1} &= \{(2,0),(2,2),(5,0),(9,0),(9,1),(12,1),(15,1)\}, \\ S^{-1} &= \{(0,2),(1,12),(4,15),(6,2),(6,5),(7,12),(8,9)\}, \\ (S \circ R)^{-1} &= \{(0,0),(0,2),(1,1),(4,1),(6,0),(6,2),(7,1),(8,0),(8,1)\}. \end{aligned}$$

We can compute $R^{-1} \circ S^{-1}$ as before and conclude that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. □

Exercise/Oppgave

2. *Let U be the universe. For the set $S \subset U$ we define the characteristic function $f_S : U \to \{0, 1\}$ by $f_S(x) := 1$ if $x \in S$ and $f_S(x) := 0$ otherwise. Now, let A and B be two sets in U. Show that $\forall x \in U$: $f_{A\cup B} = f_A(x) + f_B(x) - (f_A f_B)(x)$.*

Solution. We shall prove the formula by cases. Note that $U = (A \cup B) \cup \overline{A \cup B}$ and the union is clearly disjoint. Then we have two cases:

- $x \in \overline{A \cup B}$. This is equivalent to say that $x \notin A \cup B$, and by DeMorgan's Law, it is equivalent to say that $x \notin A$ and $x \notin B$. Since $x \notin A \cup B$, then $f_{A\cup B}(x) = 0$. On the other hand, $x \notin A$ and $x \notin B$ implies that $f_A(x) = f_B(x) = 0$, and then $f_A(x)f_B(x) = 0$. This allows us to conclude that $f_A(x) + f_B(x) - (f_A f_B)(x) = 0 = f_{A\cup B}(x)$.
- $x \in A \cup B$. In this case we have that $f_{A\cup B}(x) = 1$. We have then three disjoint subcases:
  - $x \in A$ and $x \in B$. Then $f_A(x) = 1$ and $f_B(x) = 1$. We also have that $f_A(x)f_B(x) = 1$. Hence $f_A(x) + f_B(x) - (f_A f_B)(x) = 1 + 1 - 1 = 1 = f_{A\cup B}(x)$.
  - $x \in A$ and $x \notin B$. Then $f_A(x) = 1$ and $f_B(x) = 0$. We have that $f_A(x)f_B(x) = 0$. Hence $f_A(x) + f_B(x) - (f_A f_B)(x) = 1 + 0 - 0 = 1 = f_{A\cup B}(x)$.
  - $x \notin A$ and $x \in B$. Then $f_A(x) = 0$ and $f_B(x) = 1$. We have that $f_A(x)f_B(x) = 0$. Hence $f_A(x) + f_B(x) - (f_A f_B)(x) = 0 + 1 - 0 = 1 = f_{A\cup B}(x)$.

Since in all the cases the formula holds, we conclude that

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - (f_A f_B)(x),$$

for all $x \in U$. □

Exercise/Oppgave

3. *A car dealership has 30 cars. 20 cars have radios, 8 cars have air conditioners and 25 cars have fuel injection. Note: 20 have at least two of these features and 6 have all three.*

a) How may cars have at least one of the features?
b) How many have none of these features?
c) How many have exactly one?

Solution. a) Let C be the set of cars that the dealership has. We know that $|C| = 30$. We also define

- R as the subset of cars with radios,
- A as the subset of cars with air conditioner,
- F as the subset of cars with fuel injection.

Notice that $R \cup A \cup F$ is the subset of cars that have at least one of the features. By the statement of the problem, we have that

$$|(R \cap A) \cup (R \cap F) \cup (A \cap F)| = 20 \quad \text{and} \quad |R \cap A \cap F| = 6.$$

By the Principle of Exclusion–Inclusion, we have that

$$|R \cup A \cup F| = |R| + |A| + |F| - |R \cap A| - |R \cap F| - |A \cap F| + |R \cap A \cap F|$$

Noticing that

$$|(R \cap A) \cup (R \cap F) \cup (A \cap F)| = 20 = |R \cap A| + |R \cap F| + |A \cap F| - 2|R \cap A \cap F|,$$

we get that

$$|R \cap A| + |R \cap F| + |A \cap F| = 20 + 12 = 32.$$

Hence

$$|R \cup A \cup F| = 20 + 8 + 25 - 32 + 6 = 27.$$

b) Note that the number of cars have none of the features is just $|C| - |R \cup A \cup F| = 30 - 27 = 3$.
c) The number of cars that have exactly one feature is the same that the number of cars that have at least one feature minus the number of cars that have at least two features, which is $27 - 20 = 7$. □

Exercise/Oppgave
4. Section/Sektion 6.1: *27, 46*

Solution. Already solved in Set 5. □

Exercise/Oppgave
5. Section/Sektion 6.2: *18, 20*

Solution. Already solved in Set 5. □

Exercise/Oppgave
6. Section/Sektion 6.3: *19a,b,c, 20*

Solution. • 19. a) Notice that each flip has exactly two outcomes and there are 10 independent flips. Then the total number of outcomes is $2^{10} = 1024$.

b) We need that there are exactly two heads. If this is the case, then there will be exactly 8 tails. Since the exact number of heads determines the number of tails, then we only need to count the number of ways that we can distribute the two heads into the ten flips. This number is counted by the combinations $\binom{10}{2} = 45$.

c) The number of outcomes that contain at most three tails is given by the sum of the number of outcomes that exactly contain i tails, for $i = 0, 1, 2, 3$. By the above argument, this number is given by

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = 176.$$

• 20. a) Notice that we can follow the argument from the above item. Hence, the number of bit strings with exactly three zeros is given by the number of ways that we can choose the three positions among the 10 positions of the string. This number is given by $\binom{10}{3} = 120$.

b) Having a string with more zeros than ones is equivalent to having a string with at most four ones. By the argument of 19c, this number is

$$\sum_{i=0}^{4} \binom{10}{i} = 386.$$

c) Similar to above

$$\sum_{i=7}^{10} \binom{10}{i} = 176.$$

c) Similar to above

$$\sum_{i=3}^{10} \binom{10}{i} = 968.$$

□

Exercise/Oppgave
7. Section/Sektion 6.5: *12, 14, 32, 56*

Solution. 12. Notice that we want to compute the number of combinations with repetition of taking 20 elements of five types of objects. This number is given by

$$C(20 + 6 - 1, 20) = \binom{25}{20} = 53130.$$

14. To count the number of non-negative solutions, we can think that x_1 corresponds to take x_1 objects of type 1, that x_2 corresponds to take x_2 objects of type 2, and so on. Finding a solution is equivalent to find a way to take 17 objects from four different types of objects. Hence, the number of solutions is given by

$$C(17 + 4 - 1, 17) = \binom{20}{17} = 1140.$$

32. We can see that in the word MISSISSIPPI, there are 1 M, 4 Is, 4 Ss and 2 Ps. The number of different strings that we can get is given by

$$\frac{11!}{1!4!4!2!} = 34650.$$

56. We can distribute five indistinguishable objects into three indistinguishable boxes by considering a partition of 5 of length at most 3. These partitions are the following:

$$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1).$$

Hence there are 5 ways to do the distribution. □

Exercise/Oppgave
8. Section/Sektion 6.6: *5*

Solution. We find the next larger permutation in lexicograph order after each of the following permutations:

a) 1432. The first position where $a_{i-1} < a_i$ is $a_2 = 4$. We change $a_1 = 1$ by the next larger element of the elements in front of it, in this case is 2, and then we construct the smaller permutation of the right-remaining elements. Hence the next larger permutation is 2134.

b) 54123. Following the same idea above, since $a_4 = 2 < a_5 = 3$, we change 2 by 3 and construct the smaller permutation]. The next larger permutation is 54132.
c) 12453. The idea is the same. The next larger permutation is 12534.
d) 45231. The next larger permutation is 45312.
e) 6714235. The next larger permutation is 6714253.
f) 31528764. The next larger permutation is 31542678.

□

Exercise/Oppgave
9. Section/Sektion 8.1: *11, 20*

Solution. 11. Let a_n be the number of ways to climb n stairs. For the first step (and each step), the person can climb one or two stairs. If the person climbs one stair, then it will remain $n-1$ stairs and there will be a_{n-1} ways to climb these stairs. On the other hand, if the person climbs two stairs, then it will remain $n-2$ stairs and there will be a_{n-2} ways to climb these stairs. The desired recurrence relation is the following

$$a_n = a_{n-1} + a_{n-2}, \quad \text{for } n \geq 3.$$

The initial conditions are given by a_1 and a_2. It is clear that $a_1 = 1$ since there is only one way to climb one stair, and $a_2 = 2$ since there are two ways to climb two stairs (giving one 2-stairs step or two 1-stair step). We can compute a_8 as follows:

$$a_3 = 2+1 = 3, a_4 = 3+2 = 5, a_5 = 5+3 = 8, a_6 = 8+5 = 13, a_7 = 13+8 = 21, a_8 = 21+13 = 34.$$

Hence the number of ways of the person can climb a flight of eight stairs is 34. *Note:* It is possible to define the recurrence relation from $n \geq 2$, where the initial conditions would be $a_1 = 1$ and $a_0 = 1$, considering the empty way of climbing 0 stairs. The correspondence recurrence equation and the value of a_8 are the same.

20. Let a_n be the number of ways that the bus driver can pay a toll of n cents. Recall that the order matters. For the first coin, the driver can use a nickel or a dime. If the driver uses a nickel, there will remain $n-5$ cents that he/she has to pay and this can be done of a_{n-5} ways. In the case that the driver uses a dime, there will reman $n-10$ cents that he/she has to pay and this can be done of a_{n-10} ways. The desired recurrence relation is given by:

$$a_n = a_{n-5} + a_{n-10}, \quad \text{for } n = 5k,\ k \geq 1,$$

with obvious initial conditions $a_0 = 1$ and $a_5 = 1$. Note that this is the same sequence from above, since if $n = 5k$ and $b_k = a_{5k}$, then $a_n = a_{5k} = b_k$ and $a_{n-5} + a_{n-10} = b_{k-1} + b_{k-2}$. Hence, a toll of 45 cents can be paid in b_9 ways. From above computations we get that $b_9 = b_8 + b_7 = 34 + 21 = 55$. Hence the number of ways that the driver can pay a toll of 45 cents is 55. □

Exercise/Oppgave
10. Section/Sektion 8.2: *3c,d,e,g, 6, 11, 42*

Solution.
- 3c. Solve $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$ and $a_1 = 0$. The characteristic polynomial of the recurrence is given by $x^2 - 5x + 6$. The roots of the polynomial are $x_1 = 2$

and $x_2 = 3$. The general solution is given by $a_n = a \cdot 2^n + b \cdot 3^n$. Using the initial conditions, we have

$$1 = a_0 = a \cdot 2^0 + b \cdot 3^0 = a + b, \quad 0 = a_1 = a \cdot 2^1 + b \cdot 3^1 = 2a + 3b.$$

Solving the system, we get $a = 3$ and $b = -2$. Hence the solution is given by

$$a_n = 3 \cdot 2^n - 2 \cdot 3^n, \quad \forall\, n \geq 0.$$

- 3d. Solve $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$ and $a_1 = 8$. The characteristic polynomial of the recurrence is given by $x^2 - 4x + 4$. The root of the polynomial is $x_1 = 2$ with multiplicity two. The general solution is given by $a_n = a \cdot 2^n + b \cdot 2^n n$. Using the initial conditions, we have

$$6 = a_0 = a \cdot 2^0 + b \cdot 2^0 \cdot 0 = a, \quad 8 = a_1 = a \cdot 2^1 + b \cdot 2^1 \cdot 1 = 2a + 2b.$$

  Solving the system, we get $a = 6$ and $b = -2$. Hence the solution is given by

$$a_n = 6 \cdot 2^n - 2 \cdot 2^n n, \quad \forall\, n \geq 0.$$

- 3e. Solve $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$ and $a_1 = 1$. The characteristic polynomial of the recurrence is given by $x^2 + 4x + 4$. The root of the polynomial is $x_1 = -2$ with multiplicity two. The general solution is given by $a_n = a \cdot (-2)^n + b \cdot (-2)^n n$. Using the initial conditions, we have

$$0 = a_0 = a \cdot (-2)^0 + b \cdot (-2)^0 \cdot 0 = a, \quad 1 = a_1 = a \cdot (-2)^1 + b \cdot (-2)^1 \cdot 1 = -2a - 2b.$$

  Solving the system, we get $a = 0$ and $b = -\frac{1}{2}$. Hence the solution is given by

$$a_n = (-2)^{n-1} n, \quad \forall\, n \geq 0.$$

- Solve $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$. The characteristic polynomial of the recurrence is given by $x^2 - \frac{1}{4}$ and its roots are $x_1 = \frac{1}{2}$ and $x_2 = -\frac{1}{2}$. The general solution is given by $a_n = \frac{a}{2^n} + \frac{b}{(-2)^n}$. Using the initial conditions, we have

$$1 = a_0 = \frac{a}{2^0} + \frac{b}{(-2)^0} = a + b, \quad 0 = a_1 = \frac{a}{2^1} + \frac{b}{(-2)^1} = a/2 - b/2.$$

  Solving the system, we get $a = b = \frac{1}{2}$. Hence the solution is given by

$$a_n = \frac{1}{2^{n+1}} - \frac{1}{(-2)^{n+1}}, \quad \forall\, n \geq 0.$$

- 6. First we give the corresponding recurrence relations. Let a_n be the number of messages that can be transmitted in n microseconds. For the first signal we have three different signals. If the signal 1 is used, then we have $n - 1$ microseconds remaining, and the number of messages that we can still send is a_{n-1}. If any of the other two signals are used, we have $n - 2$ microseconds remaining and the number of messages that we can still send is a_{n-2}. Hence, the desired recurrence relation is given by

$$a_n = a_{n-1} + 2a_{n-2}, \quad \forall\, n \geq 3,$$

with initial conditions $a_1 = 1$ and $a_0 = 1$. Now we solve the recurrence relation. The characteristic polynomial is $x^2 - x - 2$ whose roots are $x_1 = 2$ and $x_2 = -1$. Hence the general solution is $a_n = a \cdot 2^n + b \cdot (-1)^n$. Using the initial conditions, we have

$$1 = a_0 = a + b, \quad 1 = a_1 = 2a - b.$$

Solving the system, we have $a = 2/3$ and $b = 1/3$. Hence the solution is given by

$$a_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n, \quad \forall\, n \geq 0.$$

- 11. Define $b_n := f_{n-1} + f_{n+1}$ for all $n \geq 2$. We show that the sequence $\{b_n\}_n$ satisfies the recurrence equation defining $\{L_n\}_n$. The initial conditions are the same: $b_1 = f_0 + f_2 = 0 + 1 = 1 = L_1$ and $b_2 = f_1 + f_3 = 1 + 2 = 3 = L_0 + L_1 = L_2$. Now, for $n \geq 2$ we have that

$$\begin{aligned} b_{n-1} + b_{n-2} &= (f_{n-2} + f_n) + (f_{n-1} + f_{n-3}) \\ &= (f_{n-3} + f_{n-2}) + (f_{n-1} + f_n) \\ &= f_{n-1} + f_{n+1} \\ &= b_n, \end{aligned}$$

where we use the recurrence relation of the Fibonacci numbers. By uniqueness of the solution, we conclude that $L_n = b_n = f_{n-1} + f_{n+1}$, for all $n \geq 2$. Now, for finding an explicit formula for the Lucas numbers, we can solve the recurrence relation with initial conditions in a similar way that we did in the previous exercises. However, since we know that $L_n = f_{n-1} + f_{n+1}$ for $n \geq 2$ and

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad \forall\, n \geq 0,$$

then for $n \geq 1$ we have

$$\begin{aligned} L_n &= f_{n-1} + f_{n+1} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(1 + \left(\frac{1+\sqrt{5}}{2}\right)^2\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(1 + \left(\frac{1-\sqrt{5}}{2}\right)^2\right)\right) \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{5+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)\right) \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \end{aligned}$$

- 42. In this problem, we can define $b_n := sf_{n-1} + tf_n$ for all $n \geq 1$ and then prove that the sequence $\{b_n\}_{n\geq 1}$ satisfies the recursion $a_n = a_{n-1} + a_{n-2}$ and the corresponding initial conditions. By uniqueness, we can conclude that $a_n = b_n$ for all $n \geq 1$. Here we give a proof by finding the solutions to the recurrence. For the Fibonacci sequence $\{f_n\}_n$, we know that

$$f_n = -\frac{1}{\sqrt{5}}x_1^n + \frac{1}{\sqrt{5}}x_2^n$$

for $n \geq 0$, where

$$x_1 = \frac{1-\sqrt{5}}{2}, \quad x_2 = \frac{1+\sqrt{5}}{2}.$$

We notice that

$$x_2 - x_1 = \sqrt{5}, \quad x_1 x_2 = -1. \tag{1}$$

We will need that later. Now, we look at the recurrence relation $a_n = a_{n-1} + a_{n-2}$. The correspondence characteristic polynomial is given by $x^2 - x - 1$ which has roots given by x_1 and x_2. Then the general solution is given by $a_n = ax_1^n + bx_2^n$, for all $n \geq 0$. Using the initial conditions, we have the system

$$s = a + b, \quad t = ax_1 + bx_2.$$

Solving the system, we find

$$\begin{aligned} b &= \frac{t - sx_1}{x_2 - x_1} = \frac{t - sx_1}{\sqrt{5}} \\ a &= \frac{sx_2 - t}{x_2 - x_1} = \frac{sx_2 - t}{\sqrt{5}}. \end{aligned}$$

Hence, for $n \geq 1$ we have that

$$\begin{aligned} a_n &= ax_1^n + bx_2^n \\ &= \frac{sx_2 - t}{\sqrt{5}}x_1^n + \frac{t - sx_1}{\sqrt{5}}^n \\ &= \frac{1}{\sqrt{5}}\left(sx_2x_1^n - tx_1^n + tx_2^n - sx_1x_2^n\right) \\ \text{(using that } x_1x_2 = -1) &= s\left(\frac{1}{\sqrt{5}}x_2^{n-1} - \frac{1}{\sqrt{5}}x_1^{n-1}\right) + t\left(\frac{1}{\sqrt{5}}x_2^n - \frac{1}{\sqrt{5}}x_1^n\right) \\ &= sf_{n-1} + tf_n. \end{aligned}$$

Therefore $a_n = sf_{n-1} + tf_n$ for $n \geq 1$, as we wanted to show.

□