# TMA4140
# DISKRET MATEMATIKK – DISCRETE MATHEMATICS
# NTNU, HØST/FALL2020

Solutions Exercise Set 6

Textbook: K. H. Rosen, *Discrete Mathematics and Its Applications*, 8. edition

**Exercise/Oppgave**

**1.** *1) Write down the truth table of the so-called EXCLUSIVE OR: $p \oplus q$, which is defined to be true if either $p$ is true and $q$ is false, or $p$ is false and $q$ is true, and it is false in all other cases. Verify that $p \oplus q$ is logically equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$.*

*2) Use the laws of logic to simplify $\big(s \vee (p \wedge r \wedge s)\big) \wedge \big(p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q)\big)$.*

*3) Provide the reasons for each step (using inference rules) required to verify that the following argument is valid:*

$$\begin{array}{l} (\neg p \vee q) \Rightarrow r \\ r \Rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \Rightarrow \neg t \\ \hline \therefore p \end{array}$$

*4) Provide a specific set of truth values for $p, q, r, s$ showing that the following argument is invalid, i.e., the premises are true while the conclusion is false.*

$$\begin{array}{l} p \\ p \to r \\ p \to (q \vee \neg r) \\ \neg q \vee \neg s \\ \hline \therefore s \end{array}$$

*Solution.* 1) According to the definition, $p \oplus q$ is true if and only if $p$ and $q$ have different truth values. The truth table is the following:

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

*Date*: September 27, 2020.

Now, we proceed to write down the table truth for $p \oplus q$ and $(p \wedge \neg q) \vee (\neg p \wedge q)$:

| $p$ | $q$ | $p \oplus q$ | $\neg p$ | $\neg q$ | $p \wedge \neg q$ | $\neg p \wedge q$ | $(p \wedge \neg q) \vee (\neg p \wedge q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | F | F | F |
| T | F | T | F | T | T | F | T |
| F | T | T | T | F | F | T | T |
| F | F | F | T | T | F | F | F |

Third and eighth columns are identical, then $p \oplus q$ is logically equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$.

2) Notice that, by Absorption Law, $s \vee (p \wedge r \wedge s) \equiv s$. Also by Absorption Law, $p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q) \equiv p$. Hence

$$\big(s \vee (p \wedge r \wedge s)\big) \wedge \big(p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q)\big) \equiv s \wedge p.$$

3)

| | **Steps** | **Reasons** |
|---|---|---|
| 1 | $\neg s \wedge \neg u$ | Premise |
| 2 | $\neg u$ | Rule of Conjunctive Simplification in 1 |
| 3 | $\neg u \Rightarrow \neg t$ | Premise |
| 4 | $\neg t$ | Modus Ponens from 2 and 3 |
| 5 | $\neg s$ | Rule of Conjunctive Simplification in 1 |
| 6 | $\neg s \wedge \neg t$ | Rule of Conjunction from 4 and 5 |
| 7 | $\neg(s \vee t)$ | DeMorgan's Law in 6 |
| 8 | $r \Rightarrow (s \vee t)$ | Premise |
| 9 | $\neg r$ | Modus Tollens from 7 and 8 |
| 10 | $(\neg p \vee q) \Rightarrow r$ | Premise |
| 11 | $\neg(\neg p \vee q)$ | Modus Tollens from 9 and 10 |
| 12 | $p \wedge \neg q$ | DeMorgan's Law in 11 |
| 13 | $p$ | Rule of Conjunctive Simplification in 12 |

4) We need that the premises are true. In particular $p$ is true. Since $p \Rightarrow r$ and $p$ is true, then $r$ is true. Since $p \Rightarrow (q \vee \neg r)$ is true and $p$ is also true, then $q \vee \neg r$ is true. However, $r$ is true, then $\neg r$ is false and then $q$ is true. Finally, since $\neg q \vee \neg s$ is true and $q$ is true, then $\neg q$ is false and then $\neg s$ is true. So, $s$ is false. Then, with the values

| $p$ | $q$ | $r$ | $s$ | $p \Rightarrow r$ | $p \Rightarrow (q \vee \neg r)$ | $\neg q \vee \neg s$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T |

we have that the argument is invalid. □

**Exercise/Oppgave**

**2.** *1) Compute the power set of the set* $A := \{\{a, b\}, \{c\}, \{d, e, f\}\}$.

*2) Consider three set $A, B, C$. Recall that the symmetric difference was defined by $A \triangle B := (A - B) \cup (B - A)$. Show the following properties:*
*a) $A \triangle B = (A \cup B) - (A \cap B)$, b) $A \triangle B = B \triangle A$ and that c) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.*

*3) Consider the sets $X$ and $Y$. Show that the following statements are equivalent:*
*i) $X \subseteq Y$, ii) $X \cap Y = X$, iii) $X \cup Y = Y$.*

*4) Recall that relations are just sets, such that the set operations $\cup$, $\cap$ and complement apply to them. Let $A, B$ be two non-empty sets. Let $R \subseteq A \times B$ be a relation. We denote the domain of $R$ by $\mathrm{dom}(R)$ and the range of $R$ by $\mathrm{ran}(R)$. The complement of $R$ is defined as $\bar{R} := (A \times B) \backslash R = (A \times B) - R$. Now let $R_1, R_2 \subseteq A \times B$ be two binary relations. Show that:*
*i) $\mathrm{dom}(R_1 \cup R_2) = \mathrm{dom}(R_1) \cup \mathrm{dom}(R_2)$ ii) $\mathrm{dom}(R_1 \cap R_2) \subseteq \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2)$.*

*Solution.* 1) The power set of $A$ is defined as the collection of subsets of $A$. We have then that the power set of $A$ is the following:

$$\mathcal{P}(A) = \{\emptyset, \{\{a, b\}\}, \{\{c\}\}, \{\{d, e, f\}\}, \{\{a, b\}, \{c\}\}, \{\{c\}, \{d, e, f\}\}, \{\{a, b\}, \{d, e, f\}\}, A\}$$

2) a) We know that for $X, Y$ sets, then $X - Y = X \cap \overline{Y}$. Using this and Distributive Law, we have that

$$\begin{aligned}
A \triangle B &= (A - B) \cup (B - A) \\
&= (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) \\
&= ((A \cup B) \cap \mathcal{U}) \cap (\mathcal{U} \cap (\overline{B} \cup \overline{A})) \\
&= (A \cup B) \cap (\overline{A \cap B}) \qquad \text{(Identity Law and DeMorgan's Law)} \\
&= (A \cup B) - (A \cap B)).
\end{aligned}$$

b) This immediately follows from Commutativity Law of union of sets:

$$A \triangle B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \triangle A.$$

c)

$$\begin{aligned}
A \triangle (B \triangle C) &= (A \cap (\overline{B \triangle C})) \cup ((B \triangle C) \cap \overline{A}) && \text{(Definition of } \triangle \text{ a} \\
&= (A \cap (\overline{(B \cap \overline{C}) \cup (C \cap \overline{B})})) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{(Definition of } \triangle\text{)} \\
&= (A \cap ((\overline{B} \cup C) \cap (\overline{C} \cup B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{DeMorgan's Law} \\
&= (A \cap ((\overline{B} \cap (\overline{C} \cup B)) \cup (C \cap (\overline{C} \cup B)))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Distributive Law} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B)) \cup (C \cap \overline{C}) \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Distributive Law} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup \emptyset \cup \emptyset \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Disjoint intersect} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup (C \cap B))) \cup (((B \cap \overline{C}) \cup (C \cap \overline{B})) \cap \overline{A}) && \text{Identity Law} \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A}) && \text{Distributive Law}
\end{aligned}$$

Now, by part b), we have that $(A\triangle B)\triangle C = C\triangle(A\triangle B)$. We can use the previous development but just changing the label of the sets:

$$C\triangle(A\triangle B) = (C\cap\overline{A}\cap\overline{B})\cup(C\cap B\cap A)\cup(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{A}\cap\overline{C}).$$

Finally, because union and intersection of sets is commutative, we conclude that

$$\begin{aligned}
A\triangle(B\triangle C) &= (A\cap\overline{B}\cap\overline{C})\cup(A\cap C\cap B)\cup(B\cap\overline{C}\cap\overline{A})\cup(C\cap\overline{B}\cap\overline{A})\\
&= (A\cap\overline{B}\cap\overline{C})\cup(C\cap B\cap A)\cup(B\cap\overline{A}\cap\overline{C})\cup(C\cap\overline{A}\cap\overline{B}) \quad \text{(rearranging intersections)}\\
&= (C\cap\overline{A}\cap\overline{B})\cup(C\cap B\cap A)\cup(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{A}\cap\overline{C}) \quad \text{(rearranging unions)}\\
&= C\triangle(A\triangle B)\\
&= (A\triangle B)\triangle C,
\end{aligned}$$

that is what we wanted to prove.

3) We have to prove $i) \Leftrightarrow ii), ii) \Leftrightarrow iii)$ and $iii) \Leftrightarrow i)$. By Law of Syllogism, this is equivalent to show $i) \Rightarrow ii), ii) \Rightarrow iii)$ and $iii) \Rightarrow i)$.

Proof of $i) \Rightarrow ii)$. Assume that $X \subseteq Y$. Then $ii)$ follows by the Absorption Law. We can prove it as follows: by definition of intersection, we have that $X \cap Y \subseteq X$. On the other hand, consider $x \in X$. By $i)$, we have that $x \in Y$. By conjunction, we have that $x \in X$ and $x \in Y$, i.e., $x \in X \cap Y$. We have shown that $X \subseteq X \cap Y$. Hence $X = X \cap Y$.

Proof of $ii) \Rightarrow iii)$. Assume that $X \cap Y = X$. Then

$$X \cup Y = (X \cap Y) \cup Y = (X \cup Y) \cap (Y \cup Y) = (X \cup Y) \cap Y = Y,$$

where we used the Absorption Law (or the first implication that we proved) in the last equality since $Y \subseteq X \cup Y$.

Proof of $iii) \Rightarrow i)$. Assume now that $X \cup Y = Y$. In general, we know that $X \subseteq X \cup Y$, but since $X \cup Y = Y$, we can conclude that $X \subseteq Y$.

4) i)

$$\begin{aligned}
x \in \text{dom}(R_1 \cup R_2) &\Leftrightarrow \text{there exists } y \in B \text{ such that } (x,y) \in R_1 \cup R_2\\
&\Leftrightarrow \text{there exists } y \in B \text{ such that } (x,y) \in R_1 \text{ or } (x,y) \in R_2\\
&\Leftrightarrow x \in \text{dom}(R_1) \text{ or } x \in \text{dom}(R_2)\\
&\Leftrightarrow x \in \text{dom}(R_1) \cup \text{dom}(R_2)
\end{aligned}$$

The third ($\Leftrightarrow$) may not be so clear because in a principle, $x \in \text{dom}(R_1)$ or $x \in \text{dom}(R_2)$ is equivalent, by definition that (there exists $y \in B$ such that $(x,y) \in R$), or (there exists $y \in B$ such that $(x,y) \in R$). However, by properties of the quantifier $\exists$, the latter sentence is equivalent to say that there exists $y \in B$ such that $((x,y) \in R$ or $(x,y) \in R)$.

ii) In a similar way to $i)$, by definition of dom we have that

$$\begin{aligned} x \in \mathrm{dom}(R_1 \cap R_2) &\Leftrightarrow \text{ there exists } y \in B \text{ such that } (x,y) \in R_1 \cap R_2 \\ &\Leftrightarrow \text{ there exists } y \in B \text{ such that } (x,y) \in R_1 \text{ and } (x,y) \in R_2 \\ &\Rightarrow x \in \mathrm{dom}(R_1) \text{ and } x \in \mathrm{dom}(R_2) \\ &\Leftrightarrow x \in \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2). \end{aligned}$$

Hence $\mathrm{dom}(R_1 \cap R_2) \subseteq \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2)$. Note that in this case, we cannot guarantee the converse in the third sentence; the element $y \in B$ in the definition of dom may be different in $R_1$ and $R_2$. □

**Exercise/Oppgave**

**3.** *Define the numbers*

$$\alpha := \frac{1+\sqrt{5}}{2} \qquad \beta := \frac{1-\sqrt{5}}{2}$$

*a) Compute the numbers* $1/\alpha$ *and* $1/\beta$. *Compute the numbers* $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ *for* $n = 0,1,2,3,4,5,6$.

*b) For* $x = \alpha$ *and* $y = \beta$, *compute the values of the following functions:*

$$i)\ f(x) = x^2 - x - 1 \quad ii)\ f(y) = y^2 - y - 1 \quad iii)\ g(x,y) = xy + 1 \quad iv)\ h(x,y) = x - y - \sqrt{5}$$

$$v)\ w(x,y) = x + y - 1 \quad vi)\ v(x,y) = x^2 + y^2 - 3 \quad vii)\ u(x,y) = x^2 - y^2 - \sqrt{5}$$

*c) Use the results from part b) to show that* $2\alpha + 1 = \alpha^3 = 0$ *and* $2\beta + 1 = \beta^3 = 0$.

*d) It is known that the Fibonacci numbers* $F_n$, $n \geq 0$, *can be expressed as* $\sqrt{5}F_n = \alpha^n - \beta^n$. *Compare the numbers* $a_n$ *computed in a) with the Fibonacci numbers* $F_n$, *for* $n = 0,1,2,3,4,5,6$. *Show that*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

*e) Use the results from a)-d) together with the binomial formula,* $(a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k 1^{n-k}$, *to show by a direct calculation, that*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n}.$$

*Solution.* a) Note that

$$\frac{1}{\alpha} = \frac{2}{1+\sqrt{5}} = \frac{2}{1+\sqrt{5}} \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{2(1-\sqrt{5})}{1-5} = -\frac{1-\sqrt{5}}{2} = \beta.$$

In this way $\frac{1}{\beta} = -\alpha$. The table of the values of $a_n$ for $n = 0, \ldots, 6$ is the following:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 |

b)

- $f(\alpha) = \alpha^2 - \alpha - 1 = 0$,
- $f(\beta) = \beta^2 - \beta - 1 = 0$,
- $g(\alpha,\beta) = \alpha\beta + 1 = \alpha\left(-\frac{1}{\alpha}\right) + 1 = -1 + 1 = 0$,
- $h(\alpha,\beta) = \alpha - \beta - \sqrt{5} = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} - \sqrt{5} = \sqrt{5} - \sqrt{5} = 0$,
- $w(\alpha,\beta) = \alpha + \beta - 1 = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} - 1 = 1 - 1 = 0$,
- $v(\alpha,\beta) = \alpha^2 + \beta^2 - 3 = 0$,
- $u(\alpha,\beta) = \alpha^2 - \beta^2 - \sqrt{5} = 0$.

c) From $f(\alpha) = 0$, we have that $\alpha^2 - \alpha - 1 = 0$. Multiplying by $\alpha$ the previous equation, we get that $\alpha^3 - \alpha^2 - \alpha = 0$. Note that $f(\alpha) = 0$ also implies that $\alpha^2 = \alpha + 1$. Then

$$\alpha^3 - \alpha^2 - \alpha = 0 \Rightarrow \alpha^3 - \alpha - 1 - \alpha = 0 \Rightarrow 2\alpha + 1 - \alpha^3 = 0,$$

that is what we wanted. Since $f(\beta) = 0$, we can repeat exactly the above argument with $\beta$ instead of $\alpha$ in order to conclude that $2\beta + 1 - \beta^3 = 0$.

d) Clearly the values of $a_n$ that we got in the table of part a) are exactly the first seven Fibonacci numbers $F_n$. We know that $\sqrt{5}F_n = \alpha^n - \beta^n$, for $n \geq 0$. By item iv) in part b), we have that $\alpha - \beta = \sqrt{5}$. Hence

$$F_n = \frac{\alpha^n - \beta^n}{5} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \forall n \geq 0.$$

e) From part c), we know that $\alpha^3 = 2\alpha + 1$. Using the binomial formula, we obtain

$$\begin{aligned} \alpha^{3n} = (\alpha^3)^n &= (2\alpha + 1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (2\alpha)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^k \alpha^k. \end{aligned}$$

Since $\beta^3 = 2\beta + 1$, we also obtain that $\beta^{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k \beta^k$. Hence

$$\alpha^{3n} - \beta^{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k (\alpha^k - \beta^k).$$

Finally, dividing the previous equation by $\alpha - \beta$ and using part d), we conclude that

$$\begin{aligned} F_{3n} &= \frac{\alpha^{3n} - \beta^{3n}}{\alpha - \beta} \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^k \frac{\alpha^k - \beta^k}{\alpha - \beta} \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^k F_k. \end{aligned}$$

□

**Exercise/Oppgave**

**4.** *Define the function* $f(x) := \frac{x+1}{x-1}$ *with domain and codomain* $D := \{x \in \mathbb{R} | x \neq 1\}$. *Calculate* $(f \circ f)(x)$ *and draw a conclusion.*

*Solution.* Since the codomain of $f$ coincides with the domain of $f$, it makes sense to consider the composition $f \circ f$. If $x \in \mathbb{R} - \{1\}$ we have

$$\begin{aligned}
(f \circ f)(x) &= f(f(x)) \\
&= f\left(\frac{x+1}{x-1}\right) \\
&= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \\
&= \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-x+1}{x-1}} \\
&= \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} \\
&= \frac{2x}{2} \\
&= x.
\end{aligned}$$

Note that we can cancel $x-1$ in the sixth equality since $x-1 \neq 0$ because $x \neq 1$. We conclude that $f$ is invertible, it is its own inverse, and it is bijective. □

**Exercise/Oppgave**

**5.** *1) Show that for all positive integers $m, n$ we have the following identities:*

$$i)\ n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}. \qquad ii)\ \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

*2) We define the numbers:*

$$N(0,0) = 1, \qquad N(n,0) = 0,\ n > 0, \qquad N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},\ n \geq k \geq 1.$$

*Show that $N(n, n+1-k) = N(n,k)$. These are the Narayana numbers.*

*3) Determine the coefficient of:*

$$\begin{aligned}
i)\ xyz^2 &\quad \text{in} \quad (x+y+z)^4 \\
ii)\ xyz^2 &\quad \text{in} \quad (2x-y-z)^4 \\
iii)\ w^2x^2y^2z^2 &\quad \text{in} \quad (w+x+y+z+1)^{10}
\end{aligned}$$

*Provide a detailed argument of your way of finding the coefficients.*

*Solution.* 1) Recall the definition of binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then for the first one:

$$
\begin{aligned}
n\binom{m+n}{m} &= n\frac{(m+n)!}{m!(m+n-m)!} \\
&= n\frac{(m+n)!}{m!n!} \\
&= \frac{(m+n)!}{m!(n-1)!} \\
&= \frac{(m+1)(m+n)!}{(m+1)m!(n-1)!} \\
&= (m+1)\frac{(m+n)!}{(m+1)!(n-1)!} \\
&= (m+1)\frac{(m+n)!}{(m+1)!(m+n-(m+1))!} \\
&= (m+1)\binom{m+n}{n+1}.
\end{aligned}
$$

In a similar way, for the second one we have

$$
\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\
&= \frac{(2n)!}{n!}\left(\frac{1}{n!} - \frac{1}{(n+1)(n-1)!}\right) \\
&= \frac{(2n)!}{n!}\left(\frac{n+1}{(n+1)n!} - \frac{n}{(n+1)n(n-1)!}\right) \quad \text{completing factorial } n! = n(n-1)! \\
&= \frac{(2n)!}{n!}\left(\frac{n+1-n}{(n+1)n!}\right) \\
&= \frac{(2n)!}{n!}\left(\frac{1}{(n+1)n!}\right) \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}
$$

2) Consider $n > 0$. If $k = 0$ then $N(n,k) = 0$ and $N(n, n+1-k) = N(n, n+1) = 0$, by item (2) of the definition of Narayana numbers. Now assume that $0 < k \le n$. Then $1 \le n+1-k \le n$ and

$$\begin{aligned} N(n, n+1-k) &= \frac{1}{n}\binom{n}{n+1-k}\binom{n}{n+1-k-1} \\ &= \frac{1}{n}\binom{n}{n+1-k}\binom{n}{n-k} \\ &= \frac{1}{n}\frac{n!}{(n-k+1)!(n-(n-k+1))!}\frac{n!}{(n-k)!(n-(n-k))!} \\ &= \frac{1}{n}\frac{n!}{(n-(k-1))!(k-1)!}\frac{n!}{(n-k)!k!} \\ &= \frac{1}{n}\binom{n}{k-1}\binom{n}{k} \\ &= N(n,k). \end{aligned}$$

Hence $N(n,k) = N(n, n+1-k)$ as we wanted to show.

3) Recall the *multinomial formula:* for $m \ge 1$ and $n \ge 0$, we have that

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m},$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}.$$

i) We want to find the coefficient of $xyz^2$ in $(x+y+z)^4$. Substituting the values $n = 4$, $m = 3$, $k_1 = 1$, $k_2 = 1$, $k_3 = 2$, we get

$$\binom{4}{1,1,2} = \frac{4!}{1!1!2!} = 12.$$

ii) We have the same values that of the previous item. However, in this case we have to consider $x_1 = 2x$, $x_2 = -y$ and $x_3 = -z$. From above, 12 is the coefficient of $x_1x_2x_3^2$ in $(x_1 + x_2 + x_3)^4$. Observe that

$$x_1x_2x_3^2 = (2x)(-y)(-z)^2 = -2xyz^2.$$

Hence the coefficient of $xyz^2$ is $12 \cdot (-2) = -24$.

iii) Note that $w^2x^2y^2z^2 = w^2x^2y^2z^21^2$. Considering $n = 10$, $m = 5$, $k_i = 2$ for $i = 1,2,3,4,5$, then the coefficient of $w^2x^2y^2z^2$ in $(w+x+y+z+1)^{10}$ is

$$\binom{10}{2,2,2,2,2} = \frac{10!}{2!2!2!2!2!} = 113400.$$

□

**Exercise/Oppgave**

**6.** *Find the number of distinct permutations of the sequence of letter:*

*i) T H O S E, ii) U N U S U A L, iii) S O C I O L O G I C A L ,*

*iv) S A N N S Y N L I G H E T S T E T T H E T S F U N K S J O N E N E*

*Solution.* We recall the formula for the number of permutations of $r_k$ elements of type $k$, for $1 \leq k \leq \ell$ for some positive integer $\ell$ and $r_1 + r_2 + \cdots + r_\ell = n$:

$$\frac{n!}{r_1!r_2!\cdots r_k!}.$$

1) Since the five letters of the word THOSE are all different, the number of distinct permutations is given by $5! = 120$.

2) Since the letter U appears three times and every other letter appears one time, then the number of permutations is

$$\frac{7!}{3!1!1!1!1!} = 840.$$

3) Note that the letter O appears three times, the letter C appears two times, the letter I appears two times, the letter L appears two times, and every other letter appears one time, then the number of permutations is

$$\frac{12!}{3!2!2!2!1!1!1!} = 9979200.$$

4) Notice that there are 33 letters in the sequence such that there are

- 5 S,
- 1 A,
- 6 N,
- 1 Y,
- 1 L,
- 1 I,
- 1 G,
- 2 H,
- 5 E,
- 5 T,
- 1 F,
- 1 U,
- 1 K,
- 1 J,
- 1 O.

Hence the number of permutations is given by

$$\frac{33!}{5!1!6!1!1!1!1!2!5!5!1!1!1!1!1!} = 348963060169587773900390400 0.$$

□

**Exercise/Oppgave**

**7.** *1) The University of Bergen holds a five-a-side soccer tournament. The rules say that the members of each team must have birthdays in the same month. How many mathematics students are needed in order to guarantee that they can raise a team?*

*2) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many distinct numbers must you select from the set $A$ so as to guarantee that there are two of them that sum to 9?*

*Solution.* 1) We will prove that we need 49 mathematics students in order to guarantee a team for the soccer tournament. If we have at most 48 students, then it is possible to distribute four or less students to each month since $4 \cdot 12 = 48$. Now we see that 49 students is possible. This follows from the pigeonhole principle. Indeed, assume that we cannot have a team with 49 students. This means that for each month, we have at most four students who share the birthday in such month. Then the number of students is less or equal to $4 \cdot 12 = 48$, and this contradicts that the number of students is 49.

2) The answer is 5. If we choose at most 4, then it is possible to choose the subset $\{1, 2, 3, 4\}$, and this subset does not satisfy that there are two numbers that sum to 9. However, if we choose 5 numbers, by the pigeonhole principle, we will choose at least one of the following pairs:

$$\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\},$$

and such pair sums to 9. This is what we wanted.

□

**Exercise/Oppgave**

**8.** *Let* $Y := \{1, 2, 3, 4, \dots, 600\}$. *Use the inclusion-exclusion principle to find the numbers of positive integers in* $Y$ *that are not divisible by* 3 *or* 5 *or* 7. *Recall the definition of the floor function,* $\lfloor x \rfloor$, *which returns the greatest integer less than or equal to the real number* $x$.

*Solution.* Let $A$ be the subset of $Y$ of integers divisible by 3, $B$ be the subset of $Y$ of integers divisible by 5, and $C$ be the subset of $Y$ of integers divisible by 7. We want to compute $|Y| - |A \cup B \cup C|$. By the inclusion–exclusion principle, we we have that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Note that

- $|A| = 200$ since $\lfloor 600/3 \rfloor = 200$,
- $|B| = 120$ since $\lfloor 600/5 \rfloor = 120$,
- $|C| = 85$ since $\lfloor 600/7 \rfloor = 85$.

Also, note that $|A \cap B|$ corresponds to the subset of $Y$ of positive integers that are divisible at the same time by 3 and 5, i.e. the positive integers in $Y$ that are divisible by 15. In the same way, $|A \cap C|$ corresponds to the subset of $Y$ of positive integers that are divisible by 21, and $|B \cap C|$ corresponds to the subset of $Y$ of positive integers that are divisible by 35. Hence

- $|A \cap B| = 40$ since $\lfloor 600/15 \rfloor = 40$,
- $|A \cap C| = 28$ since $\lfloor 600/21 \rfloor = 28$,
- $|B \cap C| = 17$ since $\lfloor 600/35 \rfloor = 17$.

Finally, $|A \cap B \cap C|$ corresponds to the subset of $Y$ of positive integers that are divisible at the same time by 3, 5, and 7, i.e. the positive integers in $Y$ that are divisible by 105. Then $|A \cap B \cap C| = 5$ since $\lfloor 600/105 \rfloor = 5$. Therefore, the number of positive integers in $Y$ that are not divisible by 3 or 5 or 7 is

$$|Y| - |A \cup B \cup C| = 600 - (200 + 120 + 85 - 40 - 28 - 17 + 5) = 600 - 325 = 275.$$

□

**Exercise/Oppgave**
**9.** *What is the number of ways of arranging the six letters $A$, $E$, $M$, $O$, $U$, and $Y$ in a sequence, such that the words $ME$ and $YOU$ do not occur?*

*Solution.* Since the six letters are all different, the number of permutations without restrictions is given by $6! = 720$. Now we will subtract the number of permutations such that the words ME and YOU occur. It is easy to see that such permutations are the following:

$$MEAYOU, MEYOUA, AMEYOU, YOUMEA, YOUAME, AYOUME.$$

Hence the desired number of arrays is $720 - 6 = 714$. □

**Exercise/Oppgave**
**10.** *1) Let $a = 8316$ and $b = 10920$. Find the greatest common divisor of $a$ and $b$ and the corresponding Bézout coefficients.*

*2) Use Fermat's little theorem to compute the remainder when $3^{47}$ is divided by 23.*

*3) Let $(x_n x_{n-1} \cdots x_0)_{10}$ be the base 10 representation of the positive integer $x$. Show that $x$ is congruent $\sum_{i=0}^{n}(-1)^i x_i$ modulo 11. Use this to test whether the integer 1213141516171819 is divisible by 11.*

*4) Consider i) $6x \equiv 1 (\mathrm{mod}\ 33)$ and ii) $81x \equiv 1 (\mathrm{mod}\ 256)$. Find the solutions to these linear congruences..*

*Solution.* 1)By the Euclid's algorithm, we have that

$$\begin{aligned} 10920 &= 1 \cdot 8316 + 2604 \\ 8316 &= 3 \cdot 2604 + 504 \\ 2604 &= 5 \cdot 504 + 84 \\ 504 &= 6 \cdot 84 + 0. \end{aligned}$$

Hence $\gcd(10920, 8316) = 84$. Now, reversing the algorithm, we can find the Bézout coefficients as follows:

$$\begin{aligned} 84 &= 2604 + (-5) \cdot 504 \\ &= 2604 + (-5)(8316 + (-3) \cdot 2604) \\ &= (-5) \cdot 8316 + 16 \cdot 2604 \\ &= (-5) \cdot 8316 + 16(10920 + (-1) \cdot 8316) \\ &= 16 \cdot 10920 + (-21) \cdot 8316. \end{aligned}$$

Hence $83 = 16 \cdot 10920 + (-21) \cdot 8316$.

2) Since 23 is a prime number such that $\gcd(23, 3) = 1$, we apply Fermat's little theorem to get

$$3^{22} \equiv 1 \pmod{23}.$$

This implies

$$3^{44} \equiv (3^{22})^2 \equiv 1^2 \equiv 1 \pmod{23}.$$

Hence

$$3^{47} \equiv 3^3 \cdot 3^{44} \equiv 3^3 \cdot 1 \equiv 27 \equiv 4 \pmod{23}.$$

We conclude that the remainder when $3^{47}$ when is divided by 23 is 4.

3) Assume that $(x_n x_{n-1} \cdots x_0)_{10}$ is the base 10 representation of the positive integer $x$. In other words, we have that

$$x = x_0 \cdot 10^0 + x_1 \cdot 10^1 + \cdots + x_{n-1} \cdot 10^{n-1} + x_n \cdot 10^n = \sum_{i=0}^{n} x_i \cdot 10^i.$$

For $i \in \{0, \ldots, n\}$, consider the integer $10^i$. Notice that $10 \equiv -1 \pmod{11}$. Taking $i$-th power, we have that $10^i \equiv (-1)^i \pmod{11}$. By properties of congruences, we have that $x_i \cdot 10^i \equiv (-1)^i x_i \pmod{11}$, for any $i \in \{-, \ldots, n\}$. Adding up the congruences, we conclude that

$$x \equiv \sum_{i=0}^{n} x_i \cdot 10^i \equiv \sum_{i=0}^{n} (-1)^i x_i \pmod{11},$$

as we wanted to show.

Now, let $x = 1213141516171819$. By the above property, we have $11|x$ if and only if $11| \sum_{i=0}^{n} (-1)^i x_i$. Since

$$9 - 1 + 8 - 1 + 7 - 1 + 6 - 1 + 5 - 1 + 4 - 1 + 3 - 1 + 2 - 1 = 36,$$

and $11 \not| 36$, we conclude that $x$ is not divisible by 11.

4) i) We see that the linear congruence $6x \equiv 1 \pmod{33}$ does not have any solution. Indeed, if there is $x$ solving the congruence, we have that there exists an integer $t$ such that $6x - 1 = 33t$. Notice that $3|33$ and $3|6x$. Since $1 = 6x - 33t$, we would have that $3|1$, and this is a contradiction. Hence our initial assumption must be false, i.e., the linear congruence does not have any solution.

ii). First, we will find the inverse of 81 modulo 256. By the Euclid's algorithm, we have

$$\begin{aligned} 256 &= 3 \cdot 81 + 13 \\ 81 &= 6 \cdot 13 + 3 \\ 13 &= 4 \cdot +1 \\ 3 &= 3 \cdot 1 + 0. \end{aligned}$$

Reversing the algorithm, we have

$$\begin{aligned} 1 &= 13 + (-4) \cdot 3 \\ &= 13 + (-4)(81 + (-6) \cdot 13) \\ &= (-4) \cdot 81 + 25 \cdot 13 \\ &= (-4) \cdot 81 + 25 \cdot (256 + (-3) \cdot 81) \\ &= 25 \cdot 256 + (-79) \cdot 81. \end{aligned}$$

Hence $-79 \equiv 177 \pmod{256}$ is the inverse of 81 modulo 256. Multiplying the equation we get

$$81x \equiv 1 \pmod{256} \Leftrightarrow 177 \cdot 81x \equiv 177 \pmod{256} \Leftrightarrow x \equiv 177 \pmod{256}.$$

Hence the integers $x \equiv 177 \pmod{256}$ solve the congruence $81x \equiv 1 \pmod{256}$. □

**Exercise/Oppgave**

**11.** *Consider the function $f(n) = \cos(n) + 3$. Show that $f(n) \in \Theta(1)$.*

*Solution.* Recall the definition: $f(n) \in \Theta(g(n))$ if and only if there exist positive real numbers $C_1$ and $C_2$ and a positive real number $k$ such that $C_1|g(n)| \leq |f(n)| \leq C_2|g(n)|$, for all $n \geq k$. Note that, by definition of cosine, we have that

$$-1 \leq \cos(n) \leq 1, \qquad \forall\, n \in \mathbb{N}.$$

Adding 3, above inequality is equivalent to

$$2 \leq \cos(n) + 3 \leq 4, \ \ \forall\, n \in \mathbb{N} \quad \Leftrightarrow \quad 2 \cdot |1| \leq |\cos(n) + 3| \leq 4 \cdot |1|, \ \ \forall\, n \geq 1.$$

Taking $C_1 = 2, C_2 = 4$ and $k = 1$, we conclude that $f(n) = \cos(n) + 3$ is $\Theta(1)$, as we wanted to show. □

**Exercise/Oppgave**

**12.** *Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.*

*Solution.* We see that the characteristic polynomial of the recurrence $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$ is given by

$$p(x) = x^3 + 3x^2 + 3x + 1.$$

Next, we find the roots of the characteristic polynomial:

$$p(x) = 0 \ \Leftrightarrow \ x^3 + 3x^2 + 3x + 1 = (x+1)^3 = 0 \ \Leftrightarrow \ x = -1.$$

We see that the root $r = -1$ has multiplicity 3. Then, the solution of the recurrence is given by

$$a_n = a(-1)^n + bn(-1)^n + cn^2(-1)^n = (-1)^n(a + bn + cn^2)$$

where the constants $a, b$ and $c$ are determined by the initial conditions. We proceed to find such constants. Indeed, we have

$$\begin{aligned} 1 &= a_0 = (-1)^0(a + b \cdot 0 + c \cdot 0^2) = a, \\ -2 &= a_1 = (-1)^1(a + b \cdot 1 + c \cdot 1^2) = -a - b - c, \\ -1 &= a_2 = (-1)^2(a + b \cdot 2 + c \cdot 2^2) = a + 2b + 4c. \end{aligned}$$

Solving the above system, we find $a = 1$, $b = 3$ and $c = -2$. Hence, the solution for our recurrence is

$$a_n = (-1)^n(1 + 3n - 2n^2), \quad \forall\, n \geq 0.$$

□