TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

SOLUTIONS – EXERCISE SET 2

Exercise/Oppgave
1. Section/Sektion 2.1: *7, 12, 26*

Solution. 2.1.7.

a) Remember that the order and repetition of elements do not matter in order to define a set. So, since both sets contain the same elements, then the pair of sets are equal.
b) Notice that $1 \in \{1, \{1\}\}$ but $1 \notin \{\{1\}\}$, then the pair of sets are not equal.
c) Similarly to the previous item, $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$. Hence the pair of sets are not equal.

2.1.12. Remember that $a \in \{a\}$ and $\{a\} \subset \{a\}$. Taking $a = \emptyset$, we have the following:

a) True.
b) True.
c) False.
d) True.
e) True.
f) True.
g) True.

2.1.26.

a) No. Notice that every power set contains the empty set. Since $\emptyset \notin \emptyset$, then $\emptyset$ is not the power set of a set.
b) Yes. It is the power set of the set $\{a\}$.
c) No. Let us call $C = \{\emptyset, \{a\}, \{\emptyset, a\}\}$. If C is the power set of a set X, then $\emptyset \in X$. Hence $\{\emptyset\}$ should be an element of C, but this is not the case.
d) Yes. It is the power set of the set $\{a, b\}$.

□

Exercise/Oppgave
2. Section/Sektion 2.2: *20c,e, 36, 52*

Proof. 2.2.20.

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- c) Recall that $A - B = \{x \mid x \in A \wedge x \notin B\}$. Hence

$$\begin{aligned} x \in (A-B)-C &\Leftrightarrow x \in A-B \wedge x \notin C \\ &\Leftrightarrow x \in A \wedge x \notin B \wedge x \notin C \\ &\Rightarrow x \in A \wedge x \notin C \\ &\Leftrightarrow x \in A - C, \end{aligned}$$

  where we used that $p \wedge q \Rightarrow p$. Hence $(A-B)-C \subset A-C$.
- e)

$$\begin{aligned} x \in (B-A)\cup(C-A) &\Leftrightarrow x \in B-A \vee x \in C-A \\ &\Leftrightarrow (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A) \\ &\Leftrightarrow (x \in B \vee x \in C) \wedge x \notin A \qquad \text{(Distributive Law)} \\ &\Leftrightarrow x \in B \cup C \wedge x \notin A \\ &\Leftrightarrow x \in (B\cup C) - A. \end{aligned}$$

  Hence $(B-A)\cup(C-A) = (B\cup C)-A$.

2.2.36. By definition of cartesian product, we have

$$\begin{aligned} (x,y) \in A \times (B\cup C) &\Leftrightarrow x \in A \wedge y \in B \cup C \\ &\Leftrightarrow x \in A \wedge (y \in B \vee y \in C) \\ &\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \qquad \text{(Distributive Law)} \\ &\Leftrightarrow (x,y) \in A \times B \vee (x,y) \in A \times C \\ &\Leftrightarrow (x,y) \in (A\times B)\cup(A\times C). \end{aligned}$$

Hence $A\times(B\cup C) = (A\times B)\cup(A\times C)$. For the case of the intersection we have

$$\begin{aligned} (x,y) \in A \times (B\cap C) &\Leftrightarrow x \in A \wedge y \in B \cap C \\ &\Leftrightarrow x \in A \wedge (y \in B \wedge y \in C) \\ &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \qquad \text{(Commutative, Associative and Idempotent Law} \\ &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in A \times C \\ &\Leftrightarrow (x,y) \in (A\times B)\cap(A\times C). \end{aligned}$$

Hence $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

2.2.52. We know that $|X\cup Y| = |X| + |Y| - |X\cap Y|$. Take $X = A\cup B$ and $Y = C$ and we obtain

$$|A\cup B\cup C| = |A\cup B| + |C| - |(A\cup B)\cap C|.$$

Note that

$$|(A\cup B)\cap C| = |(A\cap C)\cup(B\cap C)| = |A\cap C| + |B\cap C| - |A\cap B\cap C|$$

and $|A\cup B| = |A| + |B| - |A\cap B|$. Substituting, we conclude

$$\begin{aligned} |A\cup B\cup C| &= |A| + |B| - |A\cap B| + |C| - (|A\cap C| + |B\cap C| - |A\cap B\cap C|) \\ &= |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|. \end{aligned}$$

□

Exercise/Oppgave

3. Section/Sektion 2.3: *12, 40, 44*

Solution. 2.3.12.

a) Yes. Notice that $f(m) = f(n) \Rightarrow m - 1 = n - 1 \Leftrightarrow m = n$.
b) No. Notice that $f(1) = 2 = f(-1)$ and $1 \neq -1$.
c) Yes. Notice that $f(m) = f(n) \Rightarrow m^3 = n^3 \Leftrightarrow m = n$ in $\mathbb{Z}$.
d) No. Since $f(1) = 1 = f(2)$, and $1 \neq 2$.

2.3.40. Note that

$$f \circ g(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b,$$

and

$$g \circ f(x) = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d.$$

Then $f \circ g = g \circ f \Leftrightarrow ad + b = bc + d \Leftrightarrow d(a - 1) = b(c - 1)$. This is our desired necessary and sufficient condition on a, b, c and d.

2.3.44. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

a) $f^{-1}(\{1\}) = \{1, -1\}$.
b) $f^{-1}(\{x \,|\, 0 < x < 1\}) = \{x \,|\, 0 < f(x) < 1\} = \{x \,|\, 0 < x < 1\} \cup \{x \,|\, -1 < x < 0\}$.
c) $f^{-1}(\{x \,|\, x > 4\}) = \{x \,|\, f(x) = x^2 > 4\} = \{x \,|\, x < -2\} \cup \{x \,|\, x > 2\}$.

□

Exercise/Oppgave

4. *We define the functions:*

(1) $$f_1 \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto f_1(x) = x + 2$$

(2) $$f_2 \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto f_2(x) = \frac{x}{x^2 + 1}$$

1) Show that f_1 is injective and surjective. Determine its inverse.
2) Prove or disprove that f_2 is injective.

Solution. 1) We have that f_1 is injective. Indeed, if $f_1(x) = f_1(y)$, then we have that $x + 2 = y + 2 \Rightarrow x = y$. Hence f_1 is injective. f_1 is also surjective. Given $x \in \mathbb{R}$, we can take $y = x - 2 \in \mathbb{R}$. Then $f_1(y) = y + 2 = x - 2 + 2 = x$. Since there exists $y \in \mathbb{R}$ such that $f_1(y) = x$, then we have that f_1 is surjective. The inverse function of f_1 is given by $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x - 2$. Then we have

$$f_1 \circ g(x) = (x - 2) + 2 = x, \quad g \circ f_1(x) = (x + 2) - 2 = x.$$

Hence g is the inverse function of f_1.

2) We can show, by algebraic manipulations, that

$$f_2(x) = f_2(y) \Rightarrow \frac{x}{x^2 + 1} = \frac{y}{y^2 + 1} \Rightarrow x = y \ \vee \ x = \frac{1}{y}.$$

Hence, since in general $x \neq 1/x$, we have that f_2 is not injective. For instance, we can take $f_2(2) = 2/5 = f_2(1/2)$, and $2 \neq 1/2$. □

Exercise/Oppgave

5. Section/Sektion 2.4: *12d, 33b,d*

Solution. 2.14.12.d Let $a_n = 2(-4)^n + 3$. Then we have

$$\begin{aligned} -3a_{n-1} + 4a_{n-2} &= -3(2(-4)^{n-1} + 3) + 4(2(-4)^{n-2} + 3) \\ &= -3 \cdot 2(-4)^{n-1} + 4 \cdot 2(-4)^{n-2} + 12 - 9 \\ &= -3 \cdot 2(-4)^{n-1} - 2(-4)^{n-1} + 3 \\ &= -2(-4)^{n-1}(3+1) + 3 \\ &= 2(-4)^n + 3 \\ &= a_n. \end{aligned}$$

Hence $a_n = 2(-4)^n + 3$ solves the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$.
2.4.33.

- b)

$$\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j) = \sum_{i=0}^{2}\sum_{j=0}^{3} 2i + \sum_{i=0}^{2}\sum_{j=0}^{3} 3j = \sum_{i=0}^{2} 8i + \sum_{i=0}^{2} 18 = 24 + 54 = 78.$$

- d)

$$\sum_{i=0}^{2}\sum_{j=1}^{3} ij = \sum_{i=0}^{2} i \sum_{j=1}^{3} j = \sum_{i=0}^{2} = 6i = 0 + 6 + 12 = 18.$$

□

Exercise/Oppgave
6. Section/Sektion 2.5: *16*

Solution. Let B be a countable set and $A \subseteq B$. We want to show that A is also countable. If B is finite, it is obvious that A is also finite, then it is countable. Otherwise, assume that B has the same cardinality as the set of positive integers. Then there exists an injection $\iota : B \to \mathbb{N}$. Consider the inclusion map $i : A \to B$. Considering the composition function, we have an injection $\iota \circ i : A \to \mathbb{N}$ since composition of injective functions is an injective as well. We conclude that A is countable as well. □

Exercise/Oppgave
7. *Use truth tables to determine which of the statements (if any) are tautologies, which are contradictions:*
1) $((p \to q) \to p) \to p$, *2)* $\neg((p \wedge \neg p) \to q)$, *3)* $p \vee (p \to \neg p)$

(1) It is a tautology, since the last column of the following table only contains 1's:

p	q	$p \to q$	$(p \to q) \to p$	$((p \to q) \to p) \to p$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	0	1

(2) It is a contradiction, since the last column of the following table only contains 0's:

p	q	$p \wedge \neg p$	$(p \wedge \neg p) \to q$	$\neg((p \wedge \neg p) \to q)$
1	1	0	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	1	0

(3) It is a tautology, since the last column of the following table only contains 1's:

p	$\neg p$	$p \to \neg p$	$p \vee (p \to \neg p)$
1	0	0	1
0	1	1	1