TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

Solutions Exercise Set 12 / Øving 12

Exercise 1. *Check whether* $(p \to (q \to r)) \to ((p \wedge q) \to r)$ *is a tautology.*

Solution. Constructing the truth table, we see that

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$	$(p \to (q \to r)) \to ((p \wedge q) \to r)$
1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	1
1	0	1	1	1	0	1	1
1	0	0	1	1	0	1	1
0	1	1	1	1	0	1	1
0	1	0	0	1	0	1	1
0	0	1	1	1	0	1	1
0	0	0	1	1	0	1	1

Since the last column of the table consists only of 1's, we conclude that $(p \to (q \to r)) \to ((p \wedge q) \to r)$ is a tautology. □

Exercise 2. *Let* X, Y, Z *be sets. Use the laws of set theory to show that*

$$(X \cup Y \cup Z) \triangle (X \cap Y \cap Z) = (X \cup Y \cup Z) \backslash (X \cap Y \cap Z).$$

Solution. Recall that by definition, $A \Delta B = (A \backslash B) \cup (B \backslash A) = (A \cap \overline{B} \cup B \cap \overline{A})$. Then we have $A \Delta B = (A \cup B) \backslash (A \cap B)$. In particular, if $B \subseteq A$ then $A \cup B = A$ and $A \cap B = B$, hence $A \Delta B = A \backslash B$. In the exercise, we clearly have that $X \cap Y \cap Z \subseteq X \cup Y \cup Z$. By above, we conclude that $(X \cup Y \cup Z) \Delta (X \cap Y \cap Z) = (X \cup Y \cup Z) \backslash (X \cap Y \cap Z)$. □

Exercise 3. *Let* $X = \{1, 2, 3, 4\}$. *Consider the relation*

$$R = \{(1,1), (1,2), (1,4), (2,3), (3,1), (3,4), (4,2), (4,4)\}$$

Determine the closures regarding: reflexive, symmetric, and transitive properties of R.

Solution.

- Reflexive closure: Recall that we have to add the pairs (a, a) where $a \in X$. Hence the reflexive closure is

$$\{(1,1), (1,2), (1,4), (2,3), (3,1), (3,4), (4,2), (4,4), (2,2), (3,3)\}.$$

- Symmetric closure: Recall that we have to add the pairs (b, a) whenever $(a, b) \in R$. Hence the symmetric closure is

$$\{(1,1), (1,2), (1,4), (2,3), (3,1), (3,4), (4,2), (4,4), (2,1), (4,1), (3,2), (1,3), (4,3), (2,4)\}.$$

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- Transitive closure: Recall that the transitive closure is given by $R^* = \bigcup_{n=1}^{|X|} R^n$. We compute:

$$\begin{aligned} R^2 &= \{(1,3),(1,2),(2,1),(2,4),(3,2),(3,4),(4,3),(1,4),(4,2),(1,1),(4,4)\} \\ R^3 &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}. \end{aligned}$$

Notice that $R^3 = X \times X \backslash \{(2,1),(2,3)\}$. However $(2,3) \in R$ and $(2,1) \in R^2$. We conclude that the transitive closure of R is $R^* = R^1 \cup R^2 \cup R^3 \cup R^4 = X \times X$.

□

Exercise 4. *i) Consider the function f with the natural numbers as domain and codomain, which maps a number written in decimal representation to the sum of squares of its decimal digits. For example, for $n = 321$, $f(321) = 3^2 + 2^2 + 1^2$. Determine whether f is injective, surjective, or both.*

ii) Show that for positive n, the function $f(n) := \sum_{i=1}^{n} i^2$ is $\Omega(n^3)$.

Solution. i) The function is not injective: indeed, for 21 and 12 we have $f(12) = 1 + 4 = 5 = 4 + 1 = f(21)$. On the other hand, f is surjective: given a natural number n, we consider the integer $11 \cdots 11$ formed by n 1's:

$$11 \cdots 1 = \sum_{i=0}^{n-1} 10^i.$$

In this case $f(11 \cdots 1) = 1^2 + 1^2 + \cdots + 1^2 = n$. Hence f is surjective.

ii) We know that $f(n) = \frac{n(n+1)(2n+1)}{6}$. Recall the definition: $f(n)$ is $\Omega(n^3)$ if and only if there are constants C and k with C positive such that $|f(n)| \geq Cn^3$ whenever $n > k$. Take $C = \frac{1}{3}$ and $k = 1$. Observe that, if $n > 1$ we have that

$$\begin{aligned} |f(n)| &= \frac{n(n+1)(2n+1)}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6} \\ &\geq \frac{2n^3}{6} \\ &= \frac{1}{3} n^3, \end{aligned}$$

where the inequality is because $3n^2 + n \geq 0$ when $n > 1$. We conclude that $f(n)$ is $\Omega(n^3)$ by using $C = \frac{1}{3}$ and $k = 1$. □

Exercise 5. *i) Use induction to show for n a positive integer that $(n+2)^{n+3} > (n+3)^{n+2}$.*

ii) Recall the definition of the Harmonic numbers H_n. Use induction to show that

$$\sum_{j=1}^{n} \frac{H_{j+1}}{j(j+1)} = 2 - \frac{1}{n+2} - \frac{H_{n+2}}{n+1}.$$

Solution. i) By mathematical induction. For the base case $n = 1$, notice that $81 = 3^4 > 4^3 = 64$, and hence the inequality is true for $n = 1$. For the inductive hypothesis, assume that $(k+2)^{k+3} > (k+3)^{k+2}$ for an integer $k \geq 1$. We will show that the inequality holds for the case $k + 1$. First, note that

$$(k+3)^2 > (k+4)(k+2).$$

This is true because $k^2+6k+9 > k^2+6k+8$. Hence

$$\begin{aligned}(k+3)^{2(k+3)} &> (k+4)^{k+3}(k+2)^{k+3}\\ &> (k+4)^{k+3}(k+3)^{k+2} \qquad \text{by inductive hypothesis.}\end{aligned}$$

Above implies that

$$(k+3)^{2k+6-k-2} > (k+4)^{k+3} \Rightarrow (k+3)^{k+4} > (k+4)^{k+3} \Rightarrow ((k+1)+2)^{(k+1)+3} > ((k+1)+3)^{(k+1)+2},$$

and this proves the case $k+1$. By mathematical induction, we conclude that $(n+2)^{n+3} > (n+3)^{n+2}$ for all $n \geq 1$.

ii) By mathematical induction. For the base case $n=1$, we have that $\sum_{j=1}^{1} \frac{H_{j+1}}{j(j+1)} = H_2/2 = 3/4$. On the other hand

$$2 - \frac{1}{3} - \frac{H_3}{3} = \frac{5}{3} - \frac{11}{12} = \frac{3}{4},$$

and so the base case is true. For the inductive hypothesis, assume that the formula holds for an integer $k \geq 1$. We will prove the formula for the case $k+1$. By splitting the sum and using the inductive hypothesis, we have that

$$\begin{aligned}\sum_{j=1}^{k+1} \frac{H_{j+1}}{j(j+1)} &= \sum_{j=1}^{k} \frac{H_{j+1}}{j(j+1)} + \frac{H_{k+2}}{(k+1)(k+2)}\\ &= 2 - \frac{1}{k+2} - \frac{H_{k+2}}{k+1} + \frac{H_{k+2}}{(k+1)(k+2)}\\ &= 2 - \frac{1}{k+2} - H_{k+2}\left(\frac{1}{k+1} - \frac{1}{(k+1)(k+2)}\right)\\ &= 2 - \frac{1}{k+2} - \frac{H_{k+2}}{k+2}\\ &= 2 - \frac{1}{k+2}\left(H_{k+2}+1\right)\\ &= 2 - \frac{1}{k+2}\left(H_{k+3}+1-\frac{1}{k+3}\right)\\ &= 2 - \frac{H_{k+3}}{k+2} + \frac{1}{k+2}\left(\frac{k+2}{k+3}\right)\\ &= 2 - \frac{1}{k+3} - \frac{H_{k+3}}{k+2}.\end{aligned}$$

Hence the inductive step is completed. By mathematical induction, we conclude that $\sum_{j=1}^{n} \frac{H_{j+1}}{j(j+1)} = 2 - \frac{1}{n+2} - \frac{H_{n+2}}{n+1}$, for all $n \geq 1$. □

Exercise 6. *Compute the coefficient of x^3 in the expansion $(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{5}})^{20}$.*

Solution. Recall the multinomial theorem:

$$(x_1 + \cdots + x_m)^n = \sum_{k_1+\cdots+k_m=n,\ k_i \geq 0} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} \cdots x_m^{k_m}.$$

Taking $m=3, n=20, x_1 = x^{\frac{1}{2}}, x_2 = x^{\frac{1}{3}}$ and $x_3 = x^{\frac{1}{5}}$, we have that

$$(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{5}})^{20} = \sum_{k_1+k_2+k_3=20,\ k_i \geq 0} \binom{20}{k_1, k_2, k_3} x^{\frac{1}{2}k_1 + \frac{1}{3}k_2 + \frac{1}{5}k_3}.$$

Hence, the coefficient of x^3 in $(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{5}})^{20}$ can be found by finding the nonnegative integers k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = 20$ and $\frac{1}{2}k_1 + \frac{1}{3}k_2 + \frac{1}{5}k_3 = 3$. The last equation is equivalent to $15k_1 + 10k_2 + 6k_3 = 90$. From the first equation, we have $k_3 = 20 - k_1 - k_2$. Substituting it into the second equation, we have

$$15k_1 + 10k_2 + 6(20 - k_1 - k_2) = 90 \quad \Rightarrow \quad 9k_1 + 4k_2 = -30.$$

However, this equation does not have solutions into the set of nonnegative integers. Hence, the previous system does not have solutions and we conclude that the term x^3 does not appear in the expression of $(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{5}})^{20}$, i.e., the coefficient of x^3 is 0. □

Exercise 7. *Find a solution to the recurrence relation $a_n = -a_{n-1} + 2a_{n-2} + 2a_{n-3}$, for $n > 3$ and $a_1 = a_2 = a_3 = 3$.*

Solution. The characteristic polynomial of the recurrence $a_n + a_{n-1} - 2a_{n-2} - 2a_{n-3} = 0$ is given by

$$x^3 + x^2 - 2x - 2 = 0$$

and has roots given by $r_1 = -1$, $r_2 = \sqrt{2}$ and $r_3 = -\sqrt{2}$. So, the general solution of the recurrence is given by

$$a_n = a(-1)^n + b(\sqrt{2})^n + c(-\sqrt{2})^n, \qquad \forall\, n \geq 1.$$

Using the initial conditions $a_1 = a_2 = a_3 = 3$ we have the system

$$\begin{aligned} 3 &= -a + b\sqrt{2} - c\sqrt{2}, \\ 3 &= a + 2b + 2c, \\ 3 &= -a + 2b\sqrt{2} - 2c\sqrt{2}, \end{aligned}$$

whose solutions are given by $a = -3$, $b = c = \frac{3}{2}$. Hence, the solution to the recurrence is

$$a_n = -3(-1)^n + \frac{3}{2}(\sqrt{2})^n + \frac{3}{2}(-\sqrt{2})^n, \qquad \forall\, n \geq 1.$$

□

Exercise 8. *Consider $\mathbb{Z}_7$ and prove that for all $[a] \in \mathbb{Z}_7$, $[a] \neq [0]$ we have that $[a]^6 = [1]$. Now, let n be a positive integer and assume that gcd$(n, 7) = 1$. Verify that 7 divides $n^6 - 1$.*

Solution. Take $[a] \neq 0$. Since $\mathbb{Z}_7$ is a field, then $[a]$ is invertible. In particular, multiplying the elements of the set $\{[1], \ldots, [6]\}$ by $[a]$, we will have that

$$\{[1], [2], \ldots, [6]\} = \{[a][1], [a][2] \ldots, [a][6]\}.$$

Hence

$$[1][2] \cdots [6] = [a][1][a][2] \cdots [a][6] = [a]^6[1][2] \cdots [6].$$

Since $[1][2] \cdots [6]$ is invertible, we conclude that $1 = [a]^6$, when $[a] \neq 0$. Now, let n be a positive integer such that $\gcd(n, 7) = 1$. Recall that, by definition $[a] \in \mathbb{Z}_7$ is the equivalence class of all the integers n such that $n \equiv a \pmod 7$. We want to show that 7 divides $n^6 - 1$, and this is equivalent to prove that $n^6 \equiv 1 \pmod 7$. Hence, we have to show that n is an integer such that $[n^6] = [1]$. Since $[n]^6 = [n^6]$ and $[n] \neq [0]$ since $7 \not| n$, we conclude by using what we proved before. □

Exercise 9. *We define for positive n the so-called prism graphs p_n. In Figure 1 you see p_4. The number T_{n+1} of spanning trees in p_{n+1} is given by $T_{n+1} = 2T_n + T_{n-1} + \cdots + T_1 + 1$. Verify this statement for $n = 4$. Show for $n > 1$ that $T_{n+1} = 3T_n - T_{n-1}$. Solve this recurrence relation for T_n, $n > 0$ and find the link to the Fibonacci numbers.*

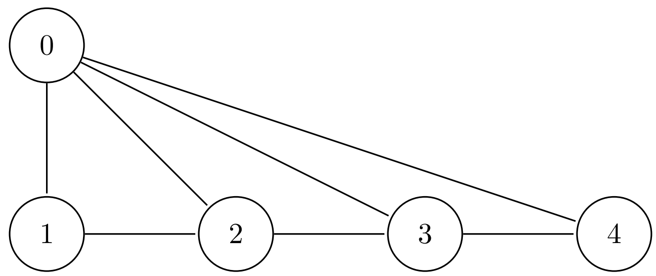


FIGURE 1. Prism graph p_4.

Solution. We can check that $T_1 = 1$, $T_2 = 3$, $T_3 = 8$, $T_4 = 21$ and $T_5 = 55$. Indeed, we can give a bijective proof by noting that any spanning tree of p_{n+1} can be given from a spanning tree of p_k for $0 \leq k \leq n$: for every spanning tree of p_n, we can give two different spanning trees of p_{n+1}, and for every spanning tree of p_k, we can give exactly one spanning tree of p_{n+1}. This reasoning leads us to the formula $T_{n+1} = 2T_n + T_{n-1} + \cdots + T_1 + 1$.

Now, we will show that $T_{n+1} = 3T_n - T_{n-1}$. Indeed,

$$\begin{aligned} T_{n+1} &= 2T_n + T_{n-1} + \cdots T_1 + 1 \\ &= 2T_n + (2T_{n-1} + T_{n-2} + \cdots + T_1 + 1) - T_{n-1} \\ &= 2T_n + T_n - T_{n-1} \\ &= 3T_n - T_{n-1}. \end{aligned}$$

Now we will solve the recursion with initial conditions $T_1 = 1$ and $T_2 = 3$. The characteristic polynomial is given by $x^2 - 3x + 1 = 0$, whose roots are $r_1 = \frac{3+\sqrt{5}}{2}$ and $r_2 = \frac{3-\sqrt{5}}{2}$. Hence, the general solution is given by

$$T_n = a\left(\frac{3+\sqrt{5}}{2}\right)^n + b\left(\frac{3-\sqrt{5}}{2}\right)^n, \qquad \forall\, n \geq 1.$$

Using the initial conditions, we have

$$\begin{aligned} 1 &= a\left(\frac{3+\sqrt{5}}{2}\right) + b\left(\frac{3-\sqrt{5}}{2}\right), \\ 3 &= a\left(\frac{3+\sqrt{5}}{2}\right)^2 + b\left(\frac{3-\sqrt{5}}{2}\right)^2. \end{aligned}$$

Solving the system, we find that $a = \frac{1}{\sqrt{5}} = -b$, and the solution to the recurrence is

$$\begin{aligned} T_n &= \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right) \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right) \qquad \forall\, n \geq 1. \end{aligned}$$

One can see that the relation with the Fibonacci number is that $T_n = F_{2n}$. □

Exercise 10. *a) i) What is the state table $T(A)$ for the automaton A in Figure 2?*

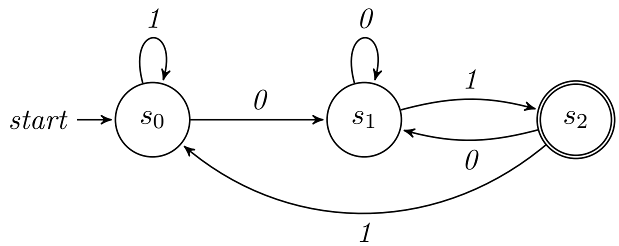


FIGURE 2. The automaton A.

ii) Determine the arrival state for each of the input sequences

$$a)\ 01 \quad b)\ 0011 \quad c)\ 010101$$

iii) What is the language $L(A)$ accepted by A?

b) *Find the language $L(A')$ accepted by the automaton A' in Fig. 3.*

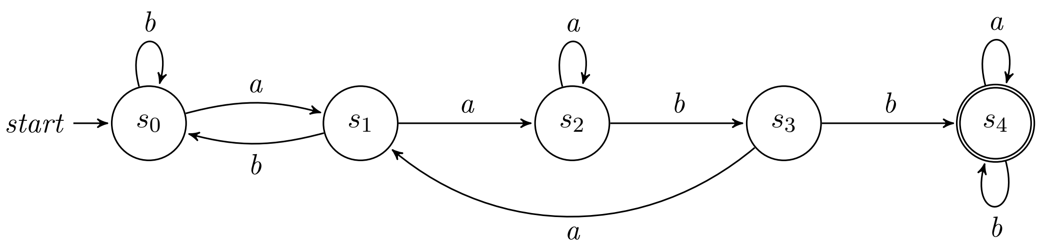


FIGURE 3. The automaton A'.

Solution. a) Consider the automaton A.

i) The state table $T(A)$ is the following:

	0	**1**
s_0	s_1	s_0
s_1	s_1	s_2
s_2	s_1	s_0

ii) a) s_2. b) s_0. c) s_2.

iii) Observe that the accepting state is s_2, and the only way to arrive to it is having an input a from s_1. On the other hand, notice that we can only arrive to s_1 by having an input 0 from any state. Hence, the language accepted by A consists in all the sequences ending with 01.

b) We find the language $L(A')$ accepted by the automaton A'. Notice that the accepting state is s_4 and, once that we reach it, we stay there. We only can reach it thought s_3 reading b, and we can only reach s_3 through s_2 reading b. Following the same idea, we see that we can only reach s_2 by reading the subword aa provided that we are not in s_4. Hence, we conclude that we can only reach s_4 by reading a word containing the subword $aabb$. This is the language accepted by the automaton A' □

Exercise 11. Section/Sektion 13.1: *Please read this section at home.*

Exercise 12. Section/Sektion 13.2: *2 a, 4 a*

Solution. 2a. The table is the following:

	0	**1**	**0**	**1**
s_0	s_1	s_2	1	0
s_1	s_0	s_3	1	0
s_2	s_3	s_0	0	0
s_3	s_1	s_2	1	1

4a. Find the output generated from the input string 10001 for the finite-state machine with the previous state diagram. The output is 00110. The successive steps are described in the following table:

Input	1	0	0	0	1	-
State	s_0	s_2	s_3	s_1	s_0	s_2
Output	0	0	1	1	0	-

□

Exercise 13. Section/Sektion 13.3: *8, 10, 12, 16, 22, 24, 36*

Solution. 8. Prove or disprove:

(1) $A \subseteq A^2$. False. If $A = \{1\}$, then $A^2 = \{11\}$ and $\{1\} \not\subseteq \{11\}$.

(2) If $A^2 = A$ then $\lambda \in A$. True. By contradiction, if $\lambda \notin A$, then let $w \in A$ be a word with minimal length. By assumption $|w| > 0$. Hence $ww \in A^2$ is a word with minimal length in A^2. Since $A = A^2$, both minimal lengths must be the same, but $|ww| \neq |w|$. Hence our original assumption is false and we conclude that $\lambda \in A$.

(3) $A\{\lambda\} = A$. True. By definition of concatenation, $A\{\lambda\}$ is the sets of strings uw such that $u \in A$ and $v \in \{\lambda\}$. Hence $v = \lambda$ and $A\{\lambda\}$ is the sets of strings $u\lambda = u$ with $u \in A$. Hence $A\{\lambda\} = A$.

(4) $(A^*)^* = A^*$. True. By definition it is clear that $A^* \subseteq \bigcup_{n=0}^{\infty}(A^*)^n = (A^*)^*$. On the other hand, consider $w \in (A^*)^*$. By definition, there exist $v_1, \ldots, v_m \in A^*$ such that $w = v_1 \cdots v_m$. Also, for every $v_j \in A^*$, there are $u_{j,1}, \ldots, u_{j,k_j}$ such that $v_j = u_{j,1} \cdots u_{j,k_j}$. Hence $w = \prod_{j=1}^{m} u_{j,1} \cdots u_{j,k_j}$, and we conclude that $w \in A^*$. Therefore $(A^*)^* \subseteq A^*$ and so $(A^*)^* = A^*$.

(5) $A^*A = A^*$. False. If $A = \{1\}$, then $A^* = \{1^n : n \geq 0\}$ and $A^*A = \{1^n : n \geq 1\} \neq A^*$.

(6) $|A^n| = |A|^n$. False. If $A = \{1, 11\}$, then $A^2 = \{11, 111, 1111\}$ and then $3 = |A^2| \neq |A|^2 = 4$.

10. Determine whether the string 01001 is in each of these sets.

(1) $\{0,1\}^*$. Yes. It is the vocabulary formed by using 0 and 1.

(2) $\{0\}^*\{10\}\{1\}^*$. No. We would have one 0, then 10, and then a string 01, but this cannot be formed in $\{1\}^*$.

(3) $\{010\}^*\{0\}^*\{1\}$. Yes. We use the substrings 010, 0 and 1.

(4) $\{010, 011\}\{00, 01\}$ Yes. We use the substrings 010 and 01.

(5) $\{00\}\{0\}^*\{01\}$. No, since 01001 does not start with 00.
(6) $\{01\}^*\{01\}^*$. No, since the substring 00 cannot be formed.

12. Determine whether each of these strings is recognized by the deterministic finite-state automaton in Figure 1. We see that s_0 and s_3 are accepting states.

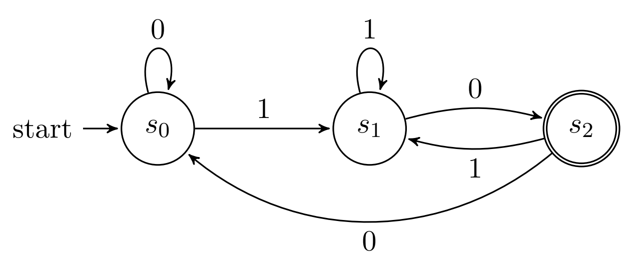
(1) 010. Yes. We have the sequence s_0, s_0, s_1, s_0.
(2) 1101. No. We have the sequence s_0, s_1, s_2, s_0, s_1.
(3) 1111110. Yes. We have the sequence $s_0, s_1, s_2, s_0, s_1, s_2, s_0, s_0$.
(4) 010101010. Yes. We have the sequence $s_0, s_0, s_1, s_0, s_1, s_0, s_1, s_0, s_1, s_0$.

16. Notice that we cannot reach s_0 once that we leave it, hence the only possibility for a word accepted by s_0 is the empty word. For s_1, notice that once that we reach it, we will stay there. We can reach it from s_0 by reading 1 as the first input, or from s_2 by reading 0. Also, we reach s_2 from s_0 only by reading 0. Hence, the language recognized by the automaton consists of the empty word, words starting with 1, and words with at least two 0's.

22. Observe that we have three accepting states: s_0, s_1 and s_5. The only possibility for being recognized by s_0 is having a word 0^n. In the same way, the only possibility for being recognized by s_1 is having a word 0^n1. Now, for s_5, we see that we can only reach it from s_3 by reading a 0. Once in s_5, we can read strings with no 0's, otherwise we would go to s_4 and that is an absorbent state (we will stay there). Also, we can reach s_3 from s_1 or s_2 by reading 0 and 1, respectively. We have seen that we can only reach s_1 from s_0, and we can reach s_2 from s_1 by reading 1. Hence, the language accepted by the automaton consists of words

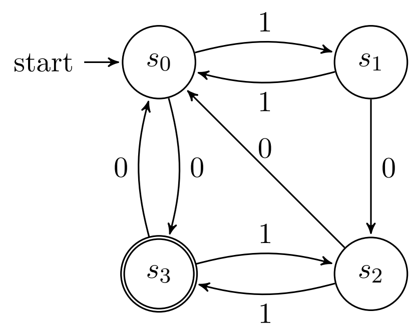
$$0^n, 0^n1, 0^n1001^m, 0^n11101^m.$$

24. Construct a deterministic finite-state automaton that recognizes the set of all bit strings that end with 10. We can give the equivalent diagram of Exercise 10a only changing the labels:



36. Construct a finite-state automaton with four states that recognizes the set of bit strings containing an even number of 1s and an odd number of 0s. We propose the following automaton.

Notice that we can reach s_3 from s_0 by reading 0, and from s_2 by reading 1. Observe that we can only reach s_2 by reading 1 from s_0 and then reading 0 from s_1, or from s_3 by reading 1. The first option will give us a contribution of two 1's and one 0's. The second will give us two 1's, and 1 0 from the step s_0 to s_3. On the other hand, if we arrive to s_3 from s_0, then we will have a contribution of one 0 (and zero 1's). The remaining cases will not change the parity of the 0's and 1's, keeping an odd number of 0's and an even number of 1's. This leads us to the desired accepted language. □


start
s_0
s_1
s_2
s_3
1
1
0
0
0
0
1
1