TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

Solutions Exercise Set 10

Exercise/Oppgave

1. *Write down the truth table for* $(p \to q) \wedge (\neg p \to r)$.

Solution. The truth table is the following

p	q	r	$\neg p$	$p \to q$	$\neg p \to r$	$(p \to q) \wedge (\neg p \to r)$
1	1	1	0	1	1	1
1	1	0	0	1	1	1
1	0	1	0	0	1	0
1	0	0	0	0	1	0
0	1	1	1	1	1	1
0	1	0	1	1	0	0
0	0	1	1	1	1	1
0	0	0	1	1	0	0

□

Exercise/Oppgave

2. *Give all steps together with reasons showing that* $(\neg p \vee \neg q) \to (p \wedge q \wedge r)$ *is logically equivalent to* $(p \wedge q)$.

Solution. Observe that

$$\begin{aligned}
(\neg p \vee \neg q) \to (p \wedge q \wedge r) &\equiv \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r) &&: \text{Definition of conditional}\\
&\equiv (p \wedge q) \vee (p \wedge q \wedge r) &&: \text{DeMorgan's Law and Double negation}\\
&\equiv (p \wedge q \wedge T) \vee (p \wedge q \wedge r) &&: s \equiv s \wedge T \text{ for a tautology } T\\
&\equiv (p \wedge q) \wedge (T \vee r) &&: \text{Distributive Law}\\
&\equiv (p \wedge q) \wedge T &&: \text{Absorption Law } T \vee r \equiv T \text{ for a tautology } T\\
&\equiv p \wedge q &&: s \equiv s \wedge T \text{ for a tautology } T.
\end{aligned}$$

□

Date: October 28, 2020.

Exercise/Oppgave

3. *Use the rules of inference to verify the following argument (give all steps together with reasons)*

$$\begin{array}{l} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \to \neg t} \\ \therefore p \end{array}$$

Solution. By the rules of inference, we have

	Step	**Reason**
1	$\neg s \wedge \neg u$	Premise
2	$\neg u$	Conjuctive Simplification from (1)
3	$\neg u \to \neg t$	Premise
4	$\neg t$	Modus Ponens from (2) and (3)
5	$\neg s$	Conjunctive Simplification from (1)
6	$\neg s \wedge \neg t$	Rule of Conjunction from (4) and (5)
7	$r \to (s \vee t)$	Premise
8	$\neg(s \vee t) \to \neg r$	Contrapositive from (7)
9	$(\neg s \wedge \neg t) \to \neg r$	DeMorgan's Law in (8)
10	$\neg r$	Modus Ponens from (6) and (9)
11	$(\neg p \vee q) \to r$	Premise
12	$\neg r \to (p \wedge \neg q)$	Contrapositive and DeMorgan's Law in (11)
13	$p \wedge \neg q$	Modus Ponens from (10) and (12)
14	p	Conjunctive Simplification from (13)

□

Exercise/Oppgave

4. *Let* $\{f_n\}_{n\geq 0}$ *be the sequence of Fibonacci numbers. Show that for* $k \geq 0$, $\gcd(f_k, f_{k+1}) = 1$.

Solution. By induction on $k \geq 0$. For the base case $k = 0$, observe that $\gcd(f_0, f_1) = \gcd(0, 1) = 1$. For the induction hypothesis, assume that there exists an integer $m \geq 0$ such that $\gcd(f_m, f_{m+1}) = 1$. We shall show that this property also happens for $m+1$. Indeed, let $d = \gcd(f_{m+1}, f_{m+2})$. We will show that $d = 1$. Indeed, since d is a common divisor of f_{m+1} and f_{m+2}, then there exist integers s, t such that $f_{m+1} = ds$ and $f_{m+2} = dt$. Also, by definition of Fibonacci numbers, we have

$$dt = f_{m+2} = f_{m+1} + f_m = ds + f_m.$$

Hence $f_m = dt - ds = d(t - s)$ and we have that d is a divisor of f_m. Since d also divides f_{m+1}, then d is a common divisor of f_m and f_{m+1}. However, by induction hypothesis, $\gcd(f_m, f_{m+1}) = 1$ implies that $d = 1$ that is what we wanted to show. □

Exercise/Oppgave

5. *Consider sets W, X, Y, Z with respect to the universal set A. Simplify $(W\cap X)\cup\big[X\cap\big((Y\cap Z)\cup(Y\cap\overline{Z})\big)\big]$. Provide all steps together with justifications from the laws of logic.*

Solution. By the laws of logic, we have that

$$
\begin{aligned}
& x\in(W\cap X)\cup\big[X\cap\big((Y\cap Z)\cup(Y\cap\overline{Z})\big)\big] & \\
\Leftrightarrow\;& x\in(W\cap X)\vee x\in\big[X\cap\big((Y\cap Z)\cup(Y\cap\overline{Z})\big)\big] & &: \text{Def. of union}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge x\in\big((Y\cap Z)\cup(Y\cap\overline{Z})\big)\big] & &: \text{Def. of intersection}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge\big(x\in(Y\cap Z)\vee x\in(Y\cap\overline{Z})\big)\big] & &: \text{Def. of union}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge\big((x\in Y\wedge x\in Z)\vee(x\in Y\wedge x\in\overline{Z})\big)\big] & &: \text{Def. of intersection}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge\big(x\in Y\wedge(x\in Z\vee x\in\overline{Z})\big)\big] & &: \text{Distributive Law}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge\big(x\in Y\wedge(x\in Z\vee x\notin Z)\big)\big] & &: \text{Def. of complement}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge\big(x\in Y\wedge x\in A\big)\big] & &: \text{Complement Law}\\
\Leftrightarrow\;& x\in(W\cap X)\vee\big[x\in X\wedge x\in Y\big] & &: \text{Identity Law}\\
\Leftrightarrow\;& (x\in W\wedge x\in X)\vee\big[x\in X\wedge x\in Y\big] & &: \text{Def. of intersection}\\
\Leftrightarrow\;& (x\in X\wedge x\in W)\vee\big[x\in X\wedge x\in Y\big] & &: \text{Commutative Law}\\
\Leftrightarrow\;& x\in X\wedge\big[x\in W\vee x\in Y\big] & &: \text{Distributive Law}\\
\Leftrightarrow\;& x\in X\wedge\big[x\in(W\cup Y)\big] & &: \text{Def. of union}\\
\Leftrightarrow\;& x\in X\cap(W\cup Y) & &: \text{Def. of intersection}
\end{aligned}
$$

Therefore $(W\cap X)\cup\big[X\cap\big((Y\cap Z)\cup(Y\cap\overline{Z})\big)\big]=X\cap(W\cup Y)$. □

Exercise/Oppgave

6. *Compute the number of solutions of the pair of equations*

$$\sum_{i=1}^{3}a_i-6=0,\quad \sum_{i=1}^{5}a_i-15=0,\quad a_i\geq 0,\; i=1,2,3,4,5.$$

Solution. Observe that the above system is equivalent to

$$a_1+a_2+a_3=6\quad\wedge\quad a_4+a_5=9,\quad a_i\geq 0,\; i=1,2,3,4,5.$$

For the first equation, notice that we can interpret a_i as the number of objects of type i, for $i=1,2,3$, and the equation is equivalent to count the number of ways that we can take 6 objects of three different type of objects, i.e. combinations with repetition. This number is given by

$$C(6+3-1,6)=\binom{8}{6}=28.$$

In a similar way, the number of solutions for the second equation is given by

$$C(9+2-1,9)=\binom{10}{9}=10.$$

Hence the number of solutions of the given pair of equations is $28\cdot 10=280$. □

Exercise/Oppgave

7. Section/Sektion 9.4: *16, 20, 24*

Solution. 16. Determine whether these sequences of vertices are paths in the directed graph:

- a, b, c, e: Yes.
- b, e, c, b, e: No. Observe that we can go from b to e but from e we can only go to d, a, or e, not to c.
- a, a, b, e, d, e: Yes.
- b, c, e, d, a, a, b: No. Observe that we have the path b, c, e, d. However, we cannot go from d to a.
- b, c, c, b, e, d, e, d: Yes.
- $a, a, b, b, c, c, b, e, d$: No. Observe that we have tha path a, a, b but we cannot go from b to b.

20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct nonstop airline flight from a to b.

- From definition $R^2 = R \circ R$. Then $(a, b) \in R^2$ if and only if there is a city c such that $(a, c) \in R$ and $(c, b) \in R$. Hence $(a, b) \in R^2$ if and only if there is a flight with exactly one intermediate stop from a to b.
- In a similar way that above, $(a, b) \in R^3$ if and only if, there is a flight with exactly two intermediate stops from a to b.
- Recall that $R^* = \bigcup_{n \geq 1} R^n$. Then $(a, b) \in R^*$ if and only if there is $n \geq 1$ such that $(a, b) \in R^n$ and this is if and only if there is $n \geq 1$ such that there is a flight with exactly $n - 1$ intermediate stops from a to b. In conclusion, $(a, b) \in R^*$ if and only if there is a flight from a to b with any number of intermediate stops.

24. Assume that a relation R is irreflexive, i.e., there is no x such that $(x, x) \in R$. We have that the relation R^2 is not necessarily irreflexive. Indeed, consider the relation

$$R = \{(1, -1), (-1, 1)\} \subset \mathbb{N} \times \mathbb{N}.$$

This is clearly an irreflexive relation. However

$$R^2 = \{(1, 1), (-1, -1)\}$$

is clearly reflexive and hence not irreflexive. □

Exercise/Oppgave

8. Section/Sektion 9.5: *9, 16*

Solution. 9. Let f be a function with domain A. Define $R \subseteq A \times A$ by $(x, y) \in R \Leftrightarrow f(x) = f(y)$. We will show that R is an equivalence relation.

- Reflexivity. Since $f(x) = f(x)$ for all $x \in A$, we clearly have that $(x, x) \in R$ for all $x \in A$.
- Symmetry. Since $f(x) = f(y) \Leftrightarrow f(y) = f(x)$, we clearly have that $(x, y) \in R \Leftrightarrow (y, x) \in R$.
- Transitivity. Assume that $(x, y), (y, z) \in R$. By definition, $f(x) = f(y)$ and $f(y) = f(z)$. This implies that $f(x) = f(z)$ and hence $(x, z) \in R$.

From above, we conclude that R is an equivalence relation. Now, for the equivalence classes, note that, if $x \in A$, the class of x is

$$[x] = \{y \in A : (x, y) \in R\} = \{y \in A : f(x) = f(y)\}.$$

Hence, the equivalence class of R are the maximal subsets of A which common image under f.

16. Define $R \subset \mathbb{N}^2 \times \mathbb{N}^2$ the relation $((a,b),(c,d)) \in R \Leftrightarrow ad = bc$. We will show that R is an equivalence relation.

- Reflexivity. Let $(a,b) \in \mathbb{N}^2$. We clearly have that $ab = ba$, so $((a,b),(a,b)) \in R$.
- Symmetry. Observe that $((a,b),(c,d)) \in R \Leftrightarrow ad = bc \Leftrightarrow cb = da \Leftrightarrow ((c,d),(a,b)) \in R$.
- Transitivity. Assume that $((a,b),(c,d)),((c,d),(e,f)) \in R$. Then we have $ad = bc$ and $cf = de$. Multiplying the last equation by a, we have $cfa = dea = ade$. Since $ad = bc$, we have $cfa = (ad)e = (bc)e$. Dividing by c, we get $af = be$. Hence $((a,b),(e,f)) \in R$.

We conclude that R is an equivalence relation. □

Exercise/Oppgave

9. Section/Sektion 9.6: *9, 18b, 27, 32*

Solution. 9. It is not a partial order since it is not transitive. To see this, notice that there are directed edges $a \to b$ and $b \to d$. If the relation is transitive, there would be a directed edge $a \to d$ but this is not the case.

18b. Recall the order of words: given two words $a = a_1 \cdots a_m$ and $b = b_1 \cdots b_n$ and $t = \min(m,n)$, we say that $a \leq b$ if $a_1 \cdots a_t \leq b_1 \cdots b_t$ (where this order is the lexicographic order), or $a_1 \cdots a_t = b_1 \cdots b_t$ and $m < n$. Hence the ordered list is the following:

open, opened, opener, opera, operand.

27. Recall that we have to consider the reflexive and transitive property. Hence the relation defined by the Hasse diagram is

$$R = \{(b,b),(a,a),(c,c),(g,g),(d,d),(e,e),(f,f),(b,g),(b,d),(b,e),(b,f),$$
$$(a,g),(a,d),(a,e),(a,f),(c,g),(c,d),(c,e),(c,f),(g,d),(g,e),(g,f)\}.$$

32.

a) The maximal elements are the vertices which do not have an outgoing edge: l and m.
b) The minimal elements are the vertices which do not have an incoming edge: a, b and c.
c) No. l and m are maximal but they are not comparable. Hence there is no a greatest element.
d) No. a and b are minimal but they are nor comparable. Hence there is no a least element.
e) The only upper bound of $\{a,b,c\}$ is l.
f) Since there is only one upper bound of $\{a,b,c\}$, then l is the least upper bound.
g) The only elements below of f are c and f. But since we cannot compare h with f nor c, then we have that there is no lower bound of $\{f,g,h\}$.
h) Since the set of lower bounds of $\{f,g,h\}$ is empty, then the greatest lower bound of $\{f,g,h\}$ does not exist.

□

Exercise/Oppgave

10. Section/Sektion 10.2: *18, 22, 26a, b, c, 57*

Solution. 18. We will prove the statement for a finite graph. Assume that $G = (V, E)$ with $|V| = n \geq 2$. By contradiction, assume that there is no pair of vertices with the same degree. Since the degree of any $v \in V$ must belong to the set $\{0, 1, \dots, n-1\}$ since the graph is simple, we must have that

$$\{\deg(v_1), \deg(v_2), \dots, \deg(v_n)\} = \{0, 1, \dots, n-1\}$$

if $V = \{v_1, v_2, \dots, v_n\}$. This implies that there exist $v, w \in V$ such that $\deg(v) = 0$ and $\deg(w) = n - 1$. However, this is a contradiction, since $\deg(v) = 0$ implies that v is isolated, and $\deg(w) = n - 1$ implies that w is connected with any other element of V, in particular with v. This contradicts with the fact that v is isolated. We conclude that there exist $v, w \in V$ with $v \neq w$ and $\deg(v) = \deg(w)$.

22. The graph is bipartite. We can apply Theorem 4 and see that by coloring a, c with blue and b, d, e with red, we will have no adjacent vertices with the same color. Therefore the bipartition is (V_1, V_2) with $V_1 = \{a, c\}$ and $V_2 = \{b, d, e\}$.

26. For which values of n are these graphs bipartite?

- K_n. For $n = 1, 2$, K_n is clearly bipartite. For $n \geq 3$, K_n is not bipartite. By contradiction, assume that (V_1, V_2) is a bipartition of K_n for a fixed $n \geq 3$. Then, by the pigeonhole principle, there is one of V_1 or V_2 with at least two different vertices v and w. However, since K_n is complete, then there is an edge between v and w, contradicting that both vertices belong to the same V_i (remember that the elements of V_i are not connected, for $i = 1, 2$). Hence K_n is not bipartite for $n \geq 3$.
- C_n. We have that C_n is bipartite if $n = 2k$ for some $k \geq 1$. Indeed, if C_{2k} is the cycle $v_1 \sim v_2 \sim v_3 \sim \cdots \sim v_{2k} \sim v_1$. Then consider $V_1 = \{v_1, v_3, \dots, v_{2k-1}\}$ and $V_2 = \{v_2, v_4, \dots, v_{2k}\}$. It is clear that the vertices in V_i are not connected, for $i = 1, 2$, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = V(C_{2k})$. Hence (V_1, V_2) is a bipartition of C_{2k}. On the other hand, C_n is not bipartite if $n = 2k + 1$ for some $k \geq 0$. By contradiction, assume that there is a bipartition (V_1, V_2) of the vertex set of $C_{2k+1} = \{v_1 \sim v_2 \sim v_3 \sim \cdots \sim v_{2k+1} \sim v_1\}$. By the pigeonhole principle, one of V_1 or V_2 has at least $k + 1$ vertices. Without loss of generality, assume that V_1 has at least $k + 1$ vertices. Notice that V_1 cannot contain all the v_i for i odd, since $v_1 \sim v_{2k+1}$. Hence, even in the extremal case that $|V_1| = k + 1$, we have that there exists j such that $v_{2j} \in V_1$. By definition of bipartition, v_{2j-1} and $v_{2j+1} \in V_2$. The first condition implies that $v_{2j-2} \in V_1$, so $v_{2j-3} \in V_2$ and so on. Repeating this argument, we have that $v_1, v_3, \dots, v_{2j-1} \in V_2$ and $v_2, \dots, v_{2j} \in V_1$. Using the same argument starting with $v_{2j+1} \in V_2$, we have that $v_{2j+1}, \dots, v_{2k+1} \in V_2$. However, this is a contradiction since $v_1, v_{2k+1} \in V_2$ and $v_1 \sim v_{2k+1}$. Therefore C_n is not bipartite for $n = 2k + 1$.
- W_n. We have that W_n is not bipartite for any $n \geq 3$. By contradiction, if W_n were bipartite with bipartition (V_1, V_2), then there exists $i \in \{1, 2\}$ such that V_i contains the central vertex v_0 of W_n. Since this vertex is connected with any other vertex of W_n, then any other vertex different than v_0 belongs to V_i, and hence $V_i = \{v_0\}$ and the other V_j contains all the remaining vertices. However,

these vertices form a cycle and clearly we have connections between vertices in V_j. This contradicts the definition of bipartition. Hence W_n is not bipartite for any $n \geq 3$.

57. How many vertices does a regular graph of degree four with 10 edges have? Let $G(V, E)$ be a regular graph of degree four with 10 edges. By the handshaking lemma, we have

$$20 = 2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V} 4 = 4|V| \quad \Rightarrow |V| = 20/4 = 5.$$

Hence the graph has 5 vertices. □