TMA4140
DISKRET MATEMATIKK – DISCRETE MATHEMATICS
NTNU, HØST/FALL2020

EXERCISE SET 6 / ØVING 6

The solutions must be submitted via OVSYS (to the assigned group/TA).
Løsningene må sendes inn via OVSYS (til den tildelte gruppen/TA).

Deadline for submission: **Friday, 9 October, 4:00pm**
Innleveringsfrist: **Fredag, 9. Oktober, kl. 16:00**

Textbook: K. H. Rosen, *Discrete Mathematics and Its Applications*, 8. edition

Exercise/Oppgave

1. *1) Write down the truth table of the so-called EXCLUSIVE OR: $p \oplus q$, which is defined to be true if either p is true and q is false, or p is false and q is true, and it is false in all other cases. Verify that $p \oplus q$ is logically equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$.*

2) Use the laws of logic to simplify $\big(s \vee (p \wedge r \wedge s)\big) \wedge \big(p \vee (p \wedge q \wedge \neg r) \vee (p \wedge q)\big)$.

3) Provide the reasons for each step (using inference rules) required to verify that the following argument is valid:

$$\begin{array}{l} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \to \neg t} \\ \therefore p \end{array}$$

4) Provide a specific set of truth values for p, q, r, s showing that the following argument is invalid, i.e., the premises are true while the conclusion is false.

$$\begin{array}{l} p \\ p \to r \\ p \to (q \vee \neg r) \\ \underline{\neg q \vee \neg s} \\ \therefore s \end{array}$$

Date: September 27, 2020.

Exercise/Oppgave

2. *1) Compute the power set of the set* $A := \{\{a,b\},\{c\},\{d,e,f\}\}$.

2) Consider three set A, B, C. *Recall that the symmetric difference was defined by* $A \triangle B := (A - B) \cup (B - A)$. *Show the following properties:*

a) $A \triangle B = (A \cup B) - (A \cap B)$, *b)* $A \triangle B = B \triangle A$ *and that c)* $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

3) Consider the sets X *and* Y. *Show that the following statements are equivalent:*

i) $X \subseteq Y$, *ii)* $X \cap Y = X$, *iii)* $X \cup Y = Y$.

4) Recall that relations are just sets, such that the set operations $\cup$, $\cap$ *and complement apply to them. Let* A, B *be two non-empty sets. Let* $R \subseteq A \times B$ *be a relation. We denote the domain of* R *by* $\mathrm{dom}(R)$ *and the range of* R *by* $\mathrm{ran}(R)$. *The complement of* R *is defined as* $\bar{R} := (A \times B) \backslash R = (A \times B) - R$. *Now let* $R_1, R_2 \subseteq A \times B$ *be two binary relations. Show that:*

i) $\mathrm{dom}(R_1 \cup R_2) = \mathrm{dom}(R_1) \cup \mathrm{dom}(R_2)$ *ii)* $\mathrm{dom}(R_1 \cap R_2) \subseteq \mathrm{dom}(R_1) \cap \mathrm{dom}(R_2)$.

Exercise/Oppgave

3. *Define the numbers*

$$\alpha := \frac{1+\sqrt{5}}{2} \qquad \beta := \frac{1-\sqrt{5}}{2}$$

a) Compute the numbers $1/\alpha$ *and* $1/\beta$. *Compute the numbers* $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ *for* $n = 0,1,2,3,4,5,6$.

b) For $x = \alpha$ *and* $y = \beta$, *compute the values of the following functions:*

$$i)\ f(x) = x^2 - x - 1 \quad ii)\ f(y) = y^2 - y - 1 \quad iii)\ g(x,y) = xy + 1 \quad iv)\ h(x,y) = x - y - \sqrt{5}$$

$$v)\ w(x,y) = x + y - 1 \quad vi)\ v(x,y) = x^2 + y^2 - 3 \quad vii)\ u(x,y) = x^2 - y^2 - \sqrt{5}$$

c) Use the results from part b) to show that $2\alpha + 1 - \alpha^3 = 0$ *and* $2\beta + 1 - \beta^3 = 0$.

d) It is known that the Fibonacci numbers F_n, $n \geq 0$, *can be expressed as* $\sqrt{5}F_n = \alpha^n - \beta^n$. *Compare the numbers* a_n *computed in a) with the Fibonacci numbers* F_n, *for* $n = 0,1,2,3,4,5,6$. *Show that*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

e) Use the results from a)-d) together with the binomial formula, $(a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k 1^{n-k}$, *to show by a direct calculation, that*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n}.$$

Exercise/Oppgave

4. *Define the function* $f(x) := \frac{x+1}{x-1}$ *with domain and codomain* $D := \{x \in \mathbb{R} | x \neq 1\}$. *Calculate* $(f \circ f)(x)$ *and draw a conclusion.*

Exercise/Oppgave

5. *1) Show that for all positive integers* m, n *we have the following identities:*

$$i)\ n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}. \qquad ii)\ \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

2) We define the numbers:

$$N(0,0)=1, \qquad N(n,0)=0,\ n>0, \qquad N(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1},\ n\geq k\geq 1.$$

Show that $N(n,n+1-k)=N(n,k)$. These are the Narayana numbers.

3) Determine the coefficient of:

$$\begin{aligned} i)\ & xyz^2 \quad \text{in} \quad (x+y+z)^4 \\ ii)\ & xyz^2 \quad \text{in} \quad (2x-y-z)^4 \\ iii)\ & w^2x^2y^2z^2 \quad \text{in} \quad (w+x+y+z+1)^{10} \end{aligned}$$

Provide a detailed argument of your way of finding the coefficients.

Exercise/Oppgave

6. *Find the number of distinct permutations of the sequence of letter:*
i) T H O S E, ii) U N U S U A L, iii) S O C I O L O G I C A L ,
iv) S A N N S Y N L I G H E T S T E T T H E T S F U N K S J O N E N E

Exercise/Oppgave

7. *1) The University of Bergen holds a five-a-side soccer tournament. The rules say that the members of each team must have birthdays in the same month. How many mathematics students are needed in order to guarantee that they can raise a team?*

2) Let $A=\{1,2,3,4,5,6,7,8\}$. How many distinct numbers must you select from the set A so as to guarantee that there are two of them that sum to 9?

Exercise/Oppgave

8. *Let $Y:=\{1,2,3,4,\ldots,600\}$. Use the inclusion-exclusion principle to find the numbers of positive integers in Y that are not divisible by 3 or 5 or 7. Recall the definition of the floor function, $\lfloor x\rfloor$, which returns the greatest integer less than or equal to the real number x.*

Exercise/Oppgave

9. *What is the number of ways of arranging the six letters A, E, M, O, U, and Y in a sequence, such that the words ME and YOU do not occur?*

Exercise/Oppgave

10. *1) Let $a=8316$ and $b=10920$. Find the greatest common divisor of a and b and the corresponding Bézout coefficients.*

2) Use Fermat's little theorem to compute the remainder when 3^{47} is divided by 23.

3) Let $(x_nx_{n-1}\cdots x_0)_{10}$ be the base 10 representation of the positive integer x. Show that x is congruent $\sum_{i=0}^{n}(-1)^ix_i$ modulo 11. Use this to test whether the integer 1213141516171819 is divisible by 11.

4) Consider i) $6x \equiv 1 (\text{mod } 33)$ *and ii)* $81x \equiv 1 (\text{mod } 256)$*. Find the solutions to these linear congruences..*

Exercise/Oppgave

11. *Consider the function* $f(n) = \cos(n) + 3$*. Show that* $f(n) \in \Theta(1)$*.*

Exercise/Oppgave

12. *Solve the recurrence relation* $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ *with initial conditions* $a_0 = 1$*,* $a_1 = -2$*, and* $a_2 = -1$*.*