10. **PROJECT. Even and Odd Functions.** **(a)** Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.
**(b)** Write $e^{kx}$, $1/(1-x)$, $\sin(x+k)$, $\cosh(x+k)$ as sums of an even and an odd function.
**(c)** Find all functions that are both even and odd.
**(d)** Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these two functions. Do you recognize familiar identities?

### Fourier Series of Even and Odd Functions

State whether the given function is even or odd. Find its Fourier series. Sketch the function and some partial sums. (Show the details of your work.)

11. $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

12. $f(x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$

13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } \pi < x < 2\pi \end{cases}$

15. $f(x) = x^2/2 \quad (-\pi < x < \pi)$

16. $f(x) = 3x(\pi^2 - x^2) \quad (-\pi < x < \pi)$

Show that

17. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$ (Use Prob. 11.)

18. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (Use Prob. 15.)

19. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{\pi^2}{12}$ (Use Prob. 15.)

### Half-Range Expansions

Find the Fourier cosine series as well as the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details.)

20. $f(x) = 1 \quad (0 < x < L)$
21. $f(x) = x \quad (0 < x < L)$
22. $f(x) = x^2 \quad (0 < x < L)$
23. $f(x) = \pi - x \quad (0 < x < \pi)$
24. $f(x) = x^3 \quad (0 < x < L)$
25. $f(x) = e^x \quad (0 < x < L)$

# 10.5 Complex Fourier Series. *Optional*

In this optional section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, below). This is done by the Euler formula (5), Sec. 2.3, with $nx$ instead of $x$, that is,

$$e^{inx} = \cos nx + i \sin nx, \tag{2}$$

$$e^{-inx} = \cos nx - i \sin nx. \tag{3}$$

By addition of these two formulas (2) and (3) and division by 2 we get

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}). \tag{4}$$

Subtraction and division by $2i$ gives

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}). \tag{5}$$

From this, using $1/i = -i$, we have in (1)

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{1}{2} a_n(e^{inx} + e^{-inx}) + \frac{1}{2i} b_n(e^{inx} - e^{-inx}) \\ &= \tfrac{1}{2}(a_n - ib_n)e^{inx} + \tfrac{1}{2}(a_n + ib_n)e^{-inx}. \end{aligned}$$

We insert this into (1), writing $a_0 = c_0$, $a_n - ib_n = c_n$, and $a_n + ib_n = k_n$. Then (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}). \tag{6}$$

For the coefficients $c_1, c_2, \cdots$ and $k_1, k_2, \cdots$ we obtain from (2), (3) and the Euler formulas (6), Sec. 10.2,

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\ k_n &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \end{aligned} \tag{7}$$

Finally, we can combine the two formulas (7) into one by the trick of writing $k_n = c_{-n}$. Then (6), (7), together with (6a) in Sec. 10.2, give

$$\boxed{\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \end{aligned}} \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{8}$$

This is the so-called **complex form of the Fourier series** or, more briefly, the **complex Fourier series**, of $f(x)$. The $c_n$ are called the **complex Fourier coefficients** of $f(x)$.

For a function of period $2L$ our reasoning gives the **complex Fourier series**

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx.} \tag{9}$$

**EXAMPLE 1 Complex Fourier series**

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

***Solution.*** Since $\sin n\pi = 0$ for integer $n$, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (8) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\frac{1}{1-in}e^{x-inx}\bigg|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n.$$

On the right,

$$\frac{1}{1-in} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2} \qquad \text{and} \qquad e^{\pi} - e^{-\pi} = 2\sinh \pi.$$

Hence the complex Fourier series is

$$e^x = \frac{\sinh \pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{1+in}{1+n^2}e^{inx} \qquad (-\pi < x < \pi). \tag{10}$$

From this let us derive the real Fourier series. Using (2) and $i^2 = -1$ we have in (10)

$$(1 + in)e^{inx} = (1 + in)(\cos nx + i \sin nx) = (\cos nx - n \sin nx) + i(n \cos nx + \sin nx).$$

Now (10) also has a corresponding term with $-n$ instead of $n$. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1 - in)e^{-inx} = (1 - in)(\cos nx - i \sin nx) = (\cos nx - n \sin nx) - i(n \cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel. Hence their sum is

$$2(\cos nx - n \sin nx), \qquad n = 1, 2, \cdots.$$

For $n = 0$ we get 1 (not 2) because there is only one term. Hence the real Fourier series is

$$e^x = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - + \cdots\right] \tag{11}$$

where $-\pi < x < \pi$. ◀

## PROBLEM SET 10.5

1. **(Calculus review)** Review complex numbers.

**Complex Fourier Series.** Find the complex Fourier series of the following functions. (Show the details of your work.)

2. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
3. $f(x) = x \quad (-\pi < x < \pi)$
4. $f(x) = 0$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
5. $f(x) = x \quad (0 < x < 2\pi)$
6. $f(x) = x^2 \quad (-\pi < x < \pi)$

7. **(Even and odd functions)** Show that the complex Fourier coefficients of an even function are real and those of an odd function are pure imaginary.
8. **(Conversion)** Convert the Fourier series in Prob. 5 to real form.
9. **(Fourier coefficients)** Show that $a_0 = c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, $n = 1, 2, \cdots$.
10. **PROJECT. Complex Fourier Coefficients.** It is very interesting that the $c_n$ in (8) can be derived directly by a method similar to that for the $a_n$ and $b_n$ in Sec. 10.2. For this, multiply the series in (8) by $e^{-imx}$ with fixed integer $m$ and integrate termwise from $-\pi$ to $\pi$ on both sides (allowed, for instance, in the case of uniform convergence), to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx.$$

Show that the integral on the right equals $2\pi$ when $n = m$ and 0 when $n \neq m$ [use (5)], so that you get the coefficient formula in (8).

# 10.6 Forced Oscillations

Fourier series have important applications in differential equations. We show this for a basic problem involving an ordinary differential equation. Numerous applications to partial differential equations will follow in Chap. 11. All this will justify Euler's and Fourier's idea of splitting up a periodic function in a series of (simpler) such functions, an idea whose enormous usefulness was far from obvious.

From Sec. 2.11 we know that forced oscillations of a body of mass $m$ on a spring of modulus $k$ are governed by the equation

$$my'' + cy' + ky = r(t); \tag{1}$$

where $y = y(t)$ is the displacement from rest, $c$ the damping constant, and $r(t)$ the external force depending on time $t$. Figure 249 shows the model and Fig. 250 its electrical analog, an $RLC$-circuit governed by

$$LI'' + RI' + \frac{1}{C}I = E'(t) \tag{1*}$$ (Sec. 2.12).

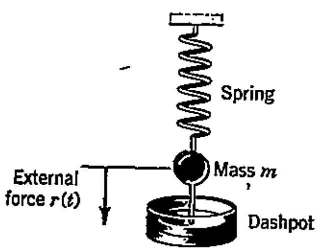


Fig. 249. Vibrating system under consideration

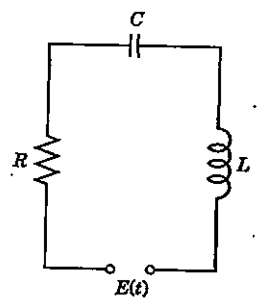


Fig. 250. Electrical analog of the system in Fig. 249 (RLC-circuit)