NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4125/30/35 Mathematics 4N/4D**

Academic contact during examination: X

Phone: Y

Examination date: -

Examination time (from–to): -

Permitted examination support material: Code C: Approved calculator
One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides)

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- There are two versions of Problem 4: one for Mathematics 4N and one for Mathematics 4D.

Language: English

Number of pages: 3

Number of pages enclosed: 2

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 [15 points]

Use the Laplace transform to solve the integral equation:

$$y(t) + 2\int_0^t y(\tau)\sin(t-\tau)\,\mathrm{d}\tau = u(t-1) - u(t-3),$$

where u is the Heaviside function (unit step function), given by

$$u(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0. \end{cases}$$

Problem 2 [10 points]

Let f be a 2π-periodic function, defined over $[-\pi, \pi]$ by

$$f(t) = \begin{cases} |x| - \pi/2 & |x| \geq \pi/2, \\ 0 & |x| < \pi/2. \end{cases}$$

Find its Fourier coefficients.

Problem 3 [10 points]

Define the convolution for functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y.$$

Use the convolution theorem to compute $f * f$, where

$$f(x) = e^{-x^2/(2a)},$$

and $a > 0$ is a constant.

Problem 4 *TMA4130 Mathematics 4N:* [6 points]

Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} x + 2 & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

Problem 4 *TMA4135 Mathematics 4D:* [6 points]

Show that the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

can be put into the form

$$\frac{\partial^2 u}{\partial y\, \partial z} = 0,$$

via the change-of-variables $y = x + t$, and $z = x - t$.

Problem 5 [14 points]

Find the solution to the following initial boundary value problem on $[0, \pi]$ using separation-of-variables:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad u(\pi, t) = 0, \quad u(x,0) = \sin(2x) + \sin(4x).$$

Problem 6 [8 points]

We will use the central difference method to discretize the second-order equation:

$$u'' + 4xu = r(x), \qquad x \in [2,5], \qquad u(2) = 3, \quad u(5) = 4.$$

Discretize the interval $[2,5]$ into intervals of length h. Let $U_i \approx u(x_i)$, and write $R_i = r(x_i)$. Write down the discrete approximation to the differential equation involving the second-order central difference of u at x_i.

Problem 7 [8 points]

a) Given the ordinary differential equation

$$y' = x^2 y, \quad y(0) = 1.$$

Write down the implicit (backward) Euler method for this equation for a given step size h.

b) Choose $h = 0.1$ and compute an approximate value for $y(0.2)$

Problem 8 [7 points]

Find the interpolation polynomial of lowest degree for the following points:

x_n	-4	0	1	2
$f(x_n)$	-85	-9	0	29

Problem 9 [10 points]

Recall the following difference formula for a four times continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$:

$$f''(a) \text{ can be approximated by } \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Assume that $|f^{(4)}(x)| \leq 1$ for all $x \in \mathbb{R}$. Use fourth order Taylor expansion to show the following error estimate

$$\Big|f''(a) - \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}\Big| \leq \frac{h^2}{12}.$$

Problem 10 [12 points]

a) Show that the equation

$$e^{-x^2} = x$$

has a unique solution on the real line.

b) Write down a bisection method for this equation, using $[0, 1]$ as the initial interval, and compute the next iteration.

c) Write down the Newton method for this equation. Compute the next iteration x_1, using $x_0 = 0.5$ as the initial point.

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\,\mathrm{d}w$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\,\mathrm{d}x$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2+a^2}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert w\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, for $n = 0, 1, 2, \ldots$, $\Gamma(n+1) = n!$
e^{at}	$\frac{1}{s-a}$
$\delta(t-a)$	e^{-as}

$$\int x^n \cos ax\,\mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax\,\mathrm{d}x$$

$$\int x^n \sin ax\,\mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\,\mathrm{d}x$$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\vec{x}_{k+1} = \vec{x}_k - JF(\vec{x}_k)^{-1}F(\vec{x}_k)$, with $JF = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$, with $l_k(x) = \prod_{j\neq k}(x - x_j)$.
- Interpolation error: $\epsilon_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0)f[x_0, x_1]+$ $(x-x_0)(x-x_1)f[x_0, x_1, x_2]+\cdots+(x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0, \dots, x_n]$, with $f[x_0, \dots, x_k] = \frac{f[x_1,\dots x_k]-f[x_0,\dots,x_{k-1}]}{x_k-x_0}$.
- Trapezoid rule: $\int_a^b f(x)\ \mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{x\in[a,b]} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\ \mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right]$.
  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{x\in[a,b]} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - \mathbf{U}\mathbf{x}^{(m)}$, with $\mathbf{A} = \mathbf{I}+\mathbf{L}+\mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(m)}$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$, $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.
- Finite differences: $\frac{\partial u}{\partial x}(x, y) \approx \frac{u(x+h,y)-u(x-h,y)}{2h}$, $\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$.
- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
  $(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.