Problem 1

a) Use Laplace convolution theorem to compute

$$f(t) := \int_0^{t+1} (t+1-x)^4 x^5 \, dx$$

The idea is to look at the following two functions

$$a(t) = t^4, \ b(t) = t^5,$$

by our definition of convolution in Laplace transform theory (different from the convolution in Fourier analysis) we have

$$(a * b)(t) = \int_0^t a(t-x)b(x) \, dx = \int_0^t (t-x)^4 x^5 \, dt.$$

Hence

$$f(t) = (a * b)(t+1).$$

Take the Laplace transform of $a * b$ and apply the Laplace convolution theorem, we get

$$\mathcal{L}(a * b) = \mathcal{L}(a) \cdot \mathcal{L}(b) = \frac{4!}{s^5} \cdot \frac{5!}{s^6} = \frac{4!5!}{10!} \frac{10!}{s^{11}},$$

together with

$$\mathcal{L}(t^{10}) = \frac{10!}{s^{11}},$$

it gives

$$(a * b)(t) = \frac{4!5!}{10!} \cdot t^{10} = \frac{t^{10}}{1260}.$$

Now we have

$$f(t) = (a * b)(t+1) = \frac{(t+1)^{10}}{1260}.$$

b) With the above f, find the solution $y(t)$ of

$$\int_0^{t+1} y(t-x) x^9 \, dx = f(t).$$

Take

$$u(t) = y(t-1), \quad v(t) = t^9,$$

as before, we have

$$\int_0^{t+1} y(t-x)x^9\,dx = \int_0^{t+1} u(t+1-x)v(x)\,dx = (u*v)(t+1).$$

By our formula for f, we have

$$(u*v)(t+1) = \frac{(t+1)^{10}}{1260},$$

hence

$$(u*v)(t) = \frac{t^{10}}{1260}.$$

Take the Laplace transform of $u*v$ and apply the Laplace convolution theorem, we get

$$\mathcal{L}(u*v) = \mathcal{L}(u)\cdot\mathcal{L}(v) = \mathcal{L}(u)\cdot\frac{9!}{s^{10}}.$$

Hence

$$\mathcal{L}(u)\cdot\frac{9!}{s^{10}} = \mathcal{L}\left(\frac{t^{10}}{1260}\right) = \frac{10!}{1260s^{11}},$$

which gives

$$\mathcal{L}(u) = \frac{1}{126s},$$

from which we get

$$u(t) = \frac{1}{126}$$

and $y(t) = 1/126$.

Problem 2

Compute the Fourier transform of

$$f(x) = \begin{cases} x+1 & |x|<1 \\ 0 & |x|\geq 1. \end{cases}$$

Input the value of f, we get

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixw}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (x+1) e^{-ixw}\, dx.$$

When $w = 0$, $\int_{-1}^{1}(x+1)e^{-ixw}\, dx = \int_{-1}^{1}(x+1)\, dx = 2$, hence $\hat{f}(w) = \sqrt{\frac{2}{\pi}}$. When $w \neq 0$, compute

$$\int_{-1}^{1} (x+1) e^{-ixw}\, dx = (x+1)\frac{e^{-ixw}}{-iw}|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-ixw}}{-iw}\, dx = \frac{2ie^{-iw}}{w} - \int_{-1}^{1} \frac{e^{-ixw}}{-iw}\, dx.$$

Compute again

$$\int_{-1}^{1} \frac{e^{-ixw}}{-iw}\, dx = \frac{e^{-ixw}}{(-iw)^2}|_{-1}^{1} = \frac{2i \sin w}{w^2},$$

we get

$$\int_{-1}^{1} (x+1) e^{-ixw}\, dx = \frac{2ie^{-iw}}{w} - \frac{2i \sin w}{w^2}.$$

To summarize, we have

$$\hat{f}(w) = \begin{cases} \sqrt{\frac{2}{\pi}} & w = 0 \\ \frac{1}{\sqrt{2\pi}} \left(\frac{2ie^{-iw}}{w} - \frac{2i \sin w}{w^2} \right) & w \neq 0. \end{cases}$$

Problem 3 *TMA4135 Mathematics 4D:*

Let us define $u(x, y) = \ln(x^2 + y^2)$ outside the origin, compute u_{xx} and u_{yy}, then show that

$$u_{xx} + u_{yy} = 0$$

outside the origin.

Notice that outside the origin we have

$$u_x = 2x/(x^2 + y^2)$$

which gives

$$u_{xx} = 2/(x^2 + y^2) - 4x^2/(x^2 + y^2)^2 = 2(y^2 - x^2)/(x^2 + y^2)^2.$$

Similarly

$$u_y = 2y/(x^2 + y^2)$$

and

$$u_{yy} = 2(x^2 - y^2)/(x^2 + y^2)^2,$$

hence

$$u_{xx} + u_{yy} = 0.$$

Problem 4 Given the partial differential equation

$$u_{tt} + 5u_{xt} + 4u_{xx} = 0 \tag{1}$$

for $u = u(x,t)$ and $x \in \mathbb{R}$ and $t > 0$.

a) Introduce new variables

$$\xi = x - 4t, \quad \eta = x - t.$$

Show that $v(\xi, \eta) = u(x,t)$ satisfies the equation

$$v_{\xi\eta} = 0.$$

b) We are given functions f and g. Show that

$$u(x,t) = \frac{1}{3}\Big(4f(x-t) - f(x-4t)\Big) + \frac{1}{3}\int_{x-4t}^{x-t} g(z)dz$$

solves equation (1) with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Solution a) By repeated use of the chain rule we find

$$\begin{aligned} u_x &= v_\xi + v_\eta, & u_{xx} &= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}, \\ u_t &= -4v_\xi - v_\eta, & u_{tt} &= 16v_{\xi\xi} + 8v_{\xi\eta} + v_{\eta\eta}, \\ u_{xt} &= -4v_{\xi\xi} - 5v_{\xi\eta} - v_{\eta\eta}. \end{aligned}$$

Inserting these expressions into equation (1), we find

$$0 = u_{tt} + 5u_{xt} + 4u_{xx} = -9v_{\xi\eta}.$$

b) With

$$u(x,t) = \frac{1}{3}\Big(4f(x-t) - f(x-4t)\Big) + \frac{1}{3}\int_{x-4t}^{x-t} g(z)dz$$

we find

$$\begin{aligned}
u_t &= \frac{1}{3}\Big(-4f'(x-t)+4f'(x-4t)\Big)+\frac{1}{3}(-g(x-t)+4g(x-4t)),\\
u_{tx} &= \frac{1}{3}\Big(-4f''(x-t)+4f''(x-4t)\Big)+\frac{1}{3}(-g'(x-t)+4g'(x-4t)),\\
u_x &= \frac{1}{3}\Big(f'(x-t)-f'(x-4t)\Big)+\frac{1}{3}(g(x-t)-g(x-4t)),\\
u_{xx} &= \frac{1}{3}\Big(f''(x-t)-f''(x-4t)\Big)+\frac{1}{3}(g'(x-t)-g'(x-4t)),
\end{aligned}$$

and hence

$$\begin{aligned}
u(x,0) &= \frac{1}{3}\Big(4f(x)-f(x)\Big)+\frac{1}{3}\int_x^x g(z)dz = f(x),\\
u_t(x,0) &= \frac{1}{3}\Big(-4f'(x)+4f'(x)\Big)+\frac{1}{3}(-g(x)+4g(x) = g(x).
\end{aligned}$$

If we insert the computed derivatives in equation (1) we find that it is satisfied.

Problem 5 Consider the heat equation

$$u_t = c^2 u_{xx} + \alpha\cos(x-\pi) \tag{2}$$

where $c > 0$ and $\alpha \in \mathbb{R}$ are given constants.

a) Show that

$$v(x,t) = \frac{\alpha}{c^2}\cos(x-\pi)$$

is a solution of (2).

b) Find the solution of equation (2) for $x \in [0,\pi]$ and $t > 0$ such that

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0,$$

and

$$u(x,0) = \frac{\alpha}{c^2}\cos(x-\pi) + \begin{cases} 0, & 0 \le x < \frac{\pi}{2},\\ x-\pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

c) Find $\lim_{t\to\infty} u(x,t)$.

Solution a) We have $v_t = 0$, and $v_{xx} = -\alpha/c^2 \cos(x-\pi)$, which implies that $v_t - c^2 v_{xx} = \alpha\cos(x-\pi)$.

b) Consider the function $w = u - v$. Linearity (or superposition) implies that w satisfies

$$w_t = c^2 w_{xx}, \quad w_x(0,t) = w_x(\pi,t) = 0, \quad t > 0,$$

(note that v_x vanishes at $x = 0$ and π), with initial data

$$w(x,0) = u(x,0) - v(x,0) = \begin{cases} 0, & 0 \le x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Standard separation of variables (Kreyszig, p. 563) gives that

$$w(x,t) = \sum_{n=0}^{\infty} A_n \cos(nx) e^{-(cn)^2 t}$$

where A_n are given as the Fourier coefficients of the initial data, thus

$$w(x,0) = \sum_{n=0}^{\infty} A_n \cos(nx) = \begin{cases} 0, & 0 \le x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Standard formulas for Fourier series yield for $n > 0$

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} \cos(nx) w(x,0)\,dx \\ &= \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos(nx)(x-\pi)\,dx \\ &= \frac{2}{\pi}\Big[\Big|_{\pi/2}^{\pi} \frac{1}{n}\sin(nx)(x-\pi) - \frac{1}{n}\int_{\pi/2}^{\pi} \sin(nx)\,dx\Big] \\ &= \frac{1}{n}\sin\Big(\frac{n\pi}{2}\Big) + \frac{2}{\pi n^2}\cos\Big(\frac{n\pi}{2}\Big). \end{aligned}$$

In addition,

$$A_0 = \frac{1}{\pi}\int_0^{\pi} w(x,0)\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} (x-\pi)\,dx = -\frac{\pi}{8}$$

Thus the answer reads

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= \frac{\alpha}{c^2}\cos(x-\pi) - \frac{\pi}{8} + \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sin\Big(\frac{n\pi}{2}\Big) + \frac{2}{\pi n^2}\cos\Big(\frac{n\pi}{2}\Big)\Big)\cos(nx) e^{-(cn)^2 t}. \end{aligned}$$

c) We see that each term in the infinite sum contains the exponentially decaying factor $e^{-(cn)^2 t}$. Thus these terms will all vanish in the limit when $t \to \infty$. Hence

$$u(x,t) \to \frac{\alpha}{c^2}\cos(x-\pi) - \frac{\pi}{8}, \quad t \to \infty.$$

Problem 6 We want to use a fixed point iteration $x_{k+1} = g(x_k)$, $k = 0, 1, \ldots$ to solve the nonlinear equation

$$0 = f(x) = \exp(x) - 2\,, \tag{3}$$

a) Find all solutions of (3) using basic calculus. Reason why these are in fact all solutions.

b) Consider the following choices for g – both given by a Python code snippet (assume that the `numpy` library is imported)

- (i)

```
def g1(x):
        return -2*(exp(x)-2)/4 + x
```

(ii)

```
def g1(x):
        return -(exp(x)-2)/4 + x
```

(iii)

```
def g1(x):
        return 2*(exp(x)-2)/4 + x
```

(iv)

```
def g1(x):
        return (exp(x)-2)/4 + x
```

-

```
def g2(x):
        return exp(x) - 2*x - 2 + 3*log(2)
```

Based on the convergence theorem for fixed point iterations (The fixed point theorem), prove or disprove in both cases whether the fixed point iteration converges towards a solution of $f(x) = 0$ if the initial guess x_0 is chosen sufficiently close to a solution.

What order of convergence do you expect in the case of `g2` and why?

c) Write down Python code that implements the according fixed point iteration function of Newton's method for problem (3).

Solution:

a) $\exp(x) - 2 = 0$ is equivalent to $x = \ln(2)$. These are all solutions, as the function $f(x) = \exp(x) - 2$ is strictly monotone.

b) The fixed point of $g1$ is $x = \ln(2)$. $x = -\frac{M\cdot(\exp(x)-2)}{4} + x$ is equivalent to $x = \ln(2)$, and so $\ln(2)$ is a fixed point of $g1$. We have $|g1'(x)| = |-\frac{M}{4}\cdot\exp(x) + 1| \overset{x=\ln(2)}{=} |-\frac{M}{2} + 1|$. We have the following variants:

- $M = -2$: $|g1'(x)| = |1 + 1| = 2 \not< 1$. So the fixed point iteration does not converge towards a solution of $f(x) = 0$.
- $M = -1$: $|g1'(x)| = |\frac{1}{2} + 1| = \frac{3}{2} \not< 1$. So the fixed point iteration does not converge towards a solution of $f(x) = 0$.
- $M = 1$: $|g1'(x)| = |-\frac{1}{2} + 1| = \frac{1}{2} < 1$. So the fixed point iteration converges towards a solution of $f(x) = 0$.
- $M = 2$: $|g1'(x)| = |-1 + 1| = 0 < 1$. So the fixed point iteration converges towards a solution of $f(x) = 0$.

We can easily see that $x = \ln(2)$ satisfies $x = g2(x)$, and therefore, $x = \ln(2)$ is a fixed point of $g2$. We have $|g2'(x)| = |\exp(x) - 2| \overset{x=\ln(2)}{=} 0 < 1$. So the fixed point iteration converges towards a solution of $f(x) = 0$.

As the value of the derivative is zero, one should locally expect (at least) second order convergence (as in the proof for convergence of Newton's method)

(c) For the above function f, the derivative is given by $f'(x) = \exp(x)$. For Newton's method, one therefore obtains

$$x_{k+1} = x_k - f(x_k)/f'(x_k) = g_{\mathbf{Newton}}(x_k) = x_k - \frac{\exp(x) - 2}{\exp(x)}$$

As Python code, this can be implemented in various ways, straightforward, it is similar to

```
def gNewton(x):
    return x - (exp(x)-2)/exp(x)
```

Problem 7 Let

$$f(x) = \begin{cases} \ln(x+1) & (x > 0) \\ \sin(x) & (x \leq 0) \end{cases}$$

and consider the nodes $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

a) What is the degree of the polynomial interpolation using the above nodes?

b) Compute the polynomial interpolation to f using the Lagrange form.

c) Determine the Newton form of the interpolating polynomial.

Solution:

a) The degree of an interpolating polynomial is at most 2.

(b) The Lagrange polynomials $\ell_i(x)$ are quadratic functions. Since $f(x_2) = 0$, we only need to consider ℓ_1 and ℓ_3

$$\ell_1(x) = \prod_{j\neq 1} \frac{x - x_j}{x_1 - x_j} = \frac{(x-0)\cdot(x-1)}{(-1)\cdot(-2)} = \frac{x^2}{2} - \frac{x}{2}$$
$$\ell_3(x) = \prod_{j\neq 3} \frac{x - x_j}{x_3 - x_j} = \frac{((x+1)\cdot(x-0)}{2\cdot 1} = \frac{x^2}{2} + \frac{x}{2}$$

The interpolating polynomial (in Lagrange form) is given by

$$p(x) = f(x_1)\cdot \ell_1(x) + f(x_3)\cdot \ell_3(x) = (-\sin 1)\cdot\left(\frac{x^2}{2} - \frac{x}{2}\right) + \ln 2\cdot\left(\frac{x^2}{2} + \frac{x^2}{2}\right).$$

(Further simplified is this identical to $\frac{\ln 2 - \sin 1}{2}\cdot x^2 + \frac{\sin 1 - + \ln 2}{2} x \approx -0.07416\cdot x^2 + 0.7673\cdot x$)

(c) Interpolating polynomial in Newton form

$$\begin{array}{llll}
x = -1 & f = -\sin 1 & & \\
 & & \sin 1 & \\
x = 0 & f = 0 & & \dfrac{-\sin 1 + \ln 2}{2} \\
 & & \ln 2 & \\
x = 1 & f = \ln 2 & &
\end{array}$$

So, we get the polynomial

$$p(x) = -\sin 1 + (x+1)\cdot\left(\sin 1 + (x-0)\cdot\frac{-\sin 1 + \ln 2}{2}\right)$$

Problem 8 Let

$$f(x) = \begin{pmatrix} \ln(x+1) & \text{for } x > 0, \\ x^2 & \text{else.} \end{pmatrix}$$

It holds $I := \int_{-1}^{1} f(x)dx = 2\cdot\ln(2) - \frac{2}{3} \approx 0.71962769445$. When using the composite Simpson's rule to approximate I, one obtains the following results:

number of subintervals m	approximation S_m	error $e_m := \lvert I - S_m \rvert$
3	0.70131264673	1.83150E-02
9	0.71757280057	2.05489E-03
27	0.71939910470	2.28590E-04
81	0.71960229224	2.54022E-05
243	0.71962487194	2.82251E-06

What order of convergence does this data suggest? Give a detailed explanation of these results on the basis of the theoretical error analysis of composites Simpson's rule from the lectures.

Solution:

The theorem from the lectures says that the composite Simpson's rule is fourth order accurate. That means that a reduction of the step size (aka a tripling of the number of steps) results (asymptotically) in a reduction of the error by a factor of $3^4 = 81$.

However, in the numerical experiment, we see the following error reductions (where in every step the number of steps is indeed tripled):

$m_1 = 3 \rightsquigarrow m_2 = 9$	error reduction $\frac{e_{m_1}}{e_{m_2}} \approx 8.9129$
$m_1 = 9 \rightsquigarrow m_2 = 27$	error reduction $\frac{e_{m_1}}{e_{m_2}} \approx 8.9894$
$m_1 = 27 \rightsquigarrow m_2 = 81$	error reduction $\frac{e_{m_1}}{e_{m_2}} \approx 8.9988$
$m_1 = 81 \rightsquigarrow m_2 = 243$	error reduction $\frac{e_{m_1}}{e_{m_2}} \approx 8.9999$

This observation corresponds to **second** order convergence (as $3^2 = 9$). The reason is that the integrand is not (as required by the theoretical result) four times continuously differentiable. Instead, one has

$$\lim_{x\to 0, x>0} f'(x) = \lim_{x\to 0, x>0} \frac{1}{1+x} = 1 \neq 0 = \lim_{x\to 0, x\le 0} 2\cdot x = \lim_{x\to 0, x\le 0} f'(x)\,.$$

Numerics of ODEs

Problem 9 We want to solve the second order ODE

$$u'' = -11u' - 10u + 2, \qquad u(0) = 1,\; u'(0) = -1,$$

using the explicit Euler method.

What is the largest step length for which we obtain a stable numerical solution?

Solution:

We start by rewriting the ODE as the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -10y_1 - 11y_2 + 2, \end{aligned}$$

or

$$\vec{y}' = A\vec{y} + \vec{b}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -10 & -11 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The Euler method is stable, if all the eigenvalues of the matrix A multiplied by the step size h lie within its stability region. Here, the eigenvalues of A are the roots of its characteristic polynomial $p(\lambda) = \lambda^2 + 11\lambda + 10$, that is, $\lambda_1 = -1$ and $\lambda_2 = -10$. The stability region of the Euler method is

$$R(z) = \{z \in \mathbb{C} : |1 + z| \le 1\}.$$

We thus arrive at the conditions

$$|1 - h| \le 1 \qquad \text{and} \qquad |1 - 10h| \le 1.$$

The first condition is equivalent to $0 \le h \le 2$, whereas the second is equivalent to $0 \le h \le 2/10$. Together, these results imply that $h = 2/10$ is the largest step size for which the solution remains stable.

Numerics of PDEs

Problem 10 Consider the two-point boundary value problem

$$u'' + x^2 u' - u = x^3 \qquad \text{for } 0 < x < 1$$

with boundary conditions

$$u(0) = 2 \qquad \text{and} \qquad u'(1) = -1$$

Set up a finite difference scheme for this problem using central differences. For the boundary condition at $x = 1$, use the idea of a false boundary and central differences. Use equidistant grid points $x_i = i\Delta x$ with a grid size $\Delta x = 1/N$.

Set up the specific system of equations for the case $N = 2$.

Solution:

We first replace the derivatives in the equation by finite differences and obtain the equation

$$\frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2} + x^2 \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} - u(x) + \mathcal{O}(\Delta x^2) = x^3.$$

Next we approximate $u_i \approx u(x_i)$ and drop the $\mathcal{O}(\Delta x^2)$ term. This results at the interior grid points x_i, $i = 1, \dots, N-1$, in the equation

$$\frac{u_{i+1} - 2u_i + u_{i+1}}{\Delta x^2} + x_i^2 \frac{u_{i+1} - u_{i-1}}{2\Delta x} - u_i = x_i^3.$$

For $i = 0$ (and $x_i = 0$) we use the Dirichlet boundary condition $u(0) = 2$, which results in the equation

$$u_0 = 2.$$

For $i = N$ (and $x_i = 1$) we use the Neumann boundary condition $u'(1) = -1$ and the idea of a false boundary. That is, we introduce the fictitious node $x_{N+1} = 1 + \Delta x$ and the corresponding fictitious function value u_{N+1}, and obtain the two equations

$$\begin{aligned} \frac{u_{N+1} - 2u_N + u_{N-1}}{\Delta x^2} + x_N^2 \frac{u_{N+1} - u_{N-1}}{2\Delta x} - u_N &= x_N^3, \\ \frac{u_{N+1} - u_{N-1}}{2\Delta x} &= -1. \end{aligned}$$

Solving the second equation for u_{N+1} yields that

$$u_{N+1} = u_{N-1} - 2\Delta x.$$

Using that $x_N = 1$, the first equation then results in

$$\frac{u_{N-1} - 2\Delta x - 2u_N + u_{N-1}}{\Delta x^2} + \frac{u_{N-1} - 2\Delta x - u_{N-1}}{2\Delta x} - u_N = 1,$$

or

$$\frac{2u_{N-1} - 2u_N}{\Delta x^2} - u_N = 2 + \frac{2}{\Delta x}.$$

For the specific case $N = 2$ and $\Delta x = 1/2$, we obtain the equations

$$\begin{aligned} u_0 &= 2, \\ 4(u_2 - 2u_1 + u_0) + \frac{1}{4}(u_2 - u_0) - u_1 &= \frac{1}{8}, \\ 8(u_1 - u_2) - u_2 &= 6. \end{aligned}$$

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} \, \mathrm{d}w$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, \mathrm{d}x$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$
$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
$\begin{cases} 1 & \text{for } \|x\| < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}t$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, for $n = 0, 1, 2, \ldots$, $\Gamma(n+1) = n!$
e^{at}	$\frac{1}{s - a}$
$\delta(t - a)$	e^{-as}

$$\int x^n \cos ax \, \mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, \mathrm{d}x$$
$$\int x^n \sin ax \, \mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, \mathrm{d}x$$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\mathbf{x}_{k+1} = \mathbf{x}_k - J(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k)$, with $J = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k)$, with $\ell_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$.
- Interpolation error: $\epsilon_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} \prod_{k=0}^{n}(x - x_k)$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0) f[x_0, x_1] +$ $(x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1}) f[x_0, \ldots, x_n]$, with $f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$.
- Trapezoidal rule: $\int_a^b f(x)\, \mathrm{d}x \approx h\left[\frac{1}{2} f(a) + f_1 + f_2 + \cdots + f_{m-1} + \frac{1}{2} f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12} h^2 \max_{x \in [a,b]} |f''(x)|$.
- Simpson's rule: $\int_a^b f(x)\, \mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{m-2} + 4f_{m-1} + f_m\right]$.
  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180} h^4 \max_{x \in [a,b]} |f^{(4)}(x)|$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler (Heun) method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2} h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,
  $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.

- Butcher tableaux for different Runge–Kutta methods:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Euler   Heun   classical RK   backward Euler

- Order conditions:

$p = 1$: $\sum_i b_i = 1$.

$p = 2$: $\sum_i b_i c_i = 1/2$.

$p = 3$: $\sum_i b_i c_i^2 = 1/3$ and $\sum_{i,j} b_i a_{ij} c_j = 1/6$.

$p = 4$: $\sum_i b_i c_i^3 = 1/4$, $\sum_{i,j} b_i c_i a_{ij} c_j = 1/8$, $\sum_{i,j} b_i a_{ij} c_j^2 = 1/12$,
and $\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = 1/24$.

- Finite differences:

$$f'(x) \approx \begin{cases} \dfrac{f(x+h) - f(x)}{h}, & \text{forward difference,} \\ \dfrac{f(x) - f(x-h)}{h}, & \text{backward difference,} \\ \dfrac{f(x+h) - f(x-h)}{2h}, & \text{central difference.} \end{cases}$$

and

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
$(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.