Problem 1 [15 points]

Use the Laplace transform to solve the integral equation:

$$y(t) + 2\int_0^t y(\tau)\sin(t-\tau)\,\mathrm{d}\tau = u(t-1) - u(t-3),$$

where u is the Heaviside function (unit step function), given by

$$u(t) = \begin{cases} 0 & t \le 0, \\ 1 & t > 0. \end{cases}$$

Solution: Applying the Laplace transform to the equation and using the convolution theorem (Laplace transform of a (properly defined) convolution is the product of Laplace transforms), we have

$$(\mathcal{L}y)(s) + 2(\mathcal{L}y)(s)\frac{1}{1+s^2} = \frac{e^{-s} - e^{-3s}}{s}.$$

Here we have also used

$$\mathcal{L}\{\sin(t)\}(s) = \frac{1}{1+s^2}, \qquad \mathcal{L}\{u(\cdot - a)\} = e^{-as}/s \quad (a > 0),$$

and the linearity of the Laplace transform.

Re-arranging the equation, we find

$$(\mathcal{L}y)(s) = \frac{s^2+1}{s^2+3}\frac{e^{-s}-e^{-3s}}{s} = \frac{e^{-s}-e^{-3s}}{s} - \frac{2}{3}\left(\frac{1}{s} - \frac{s}{s^2+3}\right)\left(e^{-s} - e^{-3s}\right).$$

The inverse transforms

$$\mathcal{L}^{-1}\left\{\frac{2}{3}\left(\frac{1}{s} - \frac{s}{s^2+3}\right)\right\}(t) = \frac{2}{3}(1 - \cos(\sqrt{3}t),$$
$$\mathcal{L}^{-1}\left\{e^{-s} - e^{-3s}\right\}(t) = \delta(t-1) - \delta(t-3),$$

hold in the appropriate senses.

Therefore

$$\begin{aligned}
y(t) &= u(t-1) - u(t-3) - \frac{2}{3}\int_0^t \Big(1-\cos(\sqrt{2}(t-\tau))\Big)\delta(\tau-1)\,\mathrm{d}\tau \\
&\quad + \frac{2}{3}\int_0^t \Big(1-\cos(\sqrt{2}(t-\tau))\Big)\delta(\tau-3)\,\mathrm{d}\tau \\
&= u(t-1) - u(t-3) - \begin{cases} 0 & 0<t<1 \\ \frac{2}{3}\Big(1-\cos(\sqrt{3}(t-1))\Big) & 1<t<3 \\ \frac{2}{3}\Big(\cos(\sqrt{3}(t-3)) - \cos(\sqrt{3}(t-1))\Big) & t>3 \end{cases} \\
&= \begin{cases} 0 & 0<t<1 \\ 1-\frac{2}{3}\Big(1-\cos(\sqrt{2}(t-1))\Big) & 1<t<3 \\ \frac{2}{3}\Big(\cos(\sqrt{2}(t-1)) - \cos(\sqrt{2}(t-3))\Big) & t>3 \end{cases}.
\end{aligned}$$

Problem 2 [10 points]

Let f be a 2π-periodic function, defined over $[-\pi, \pi]$ by

$$f(t) = \begin{cases} |x| - \pi/2 & |x| \geq \pi/2, \\ 0 & |x| < \pi/2. \end{cases}$$

Find the real (or complex) Fourier coefficients of f.

Solution: Notice that f is an even function. Therefore the sine coefficients b_n are all zero. And from Chapter 11.2 of Kreyszig 10th ed, the cosine coefficients for $n > 0$ are

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{\pi/2}^\pi (x-\pi/2)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{n\pi}(x-\pi/2)\sin(nx)\Big|_{\pi/2}^{\pi} - \frac{2}{n\pi}\int_{\pi/2}^\pi \sin(nx)\,\mathrm{d}x \\
&= 0 + \frac{2}{n^2\pi}\cos(nx)\Big|_{\pi/2}^{\pi} \\
&= \frac{2}{n^2\pi}\Big(\cos(n\pi) - \cos(n\pi/2)\Big),
\end{aligned}$$

which can be expressed as

$$= \frac{2}{n^2\pi} \cdot \begin{cases} 1-1 & n/4 \in \mathbb{Z} \\ 1-(-1) & (n-2)/4 \in \mathbb{Z} \\ -1-0 & (n-1)/2 \in \mathbb{Z} \end{cases}$$
$$= \frac{2}{n^2\pi} \cdot \begin{cases} 0 & n/4 \in \mathbb{Z} \\ 2 & (n-2)/4 \in \mathbb{Z} \\ -1 & (n-1)/2 \in \mathbb{Z} \end{cases}.$$

and the 0th coefficient is the average over the period of length 2π, which is $(\pi/2)^2/(2\pi) = a_0$, by the formula for the area of a triangle.
Therefore the complex coefficients are $c_0 = a_0$, and

$$c_n = (a_{|n|} - \operatorname{sgn}(n) i b_{|n|})/2 = a_{|n|}/2 = \frac{1}{n^2\pi} \cdot \begin{cases} 0 & n/4 \in \mathbb{Z} \\ 2 & (n-2)/4 \in \mathbb{Z} \\ -1 & (n-1)/2 \in \mathbb{Z} \end{cases}, \quad n \neq 0.$$

Problem 3 [10 points]

Define the convolution for functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, \mathrm{d}y.$$

Use the convolution theorem to compute $f * f$, where

$$f(x) = e^{-x^2/(2a)},$$

and $a > 0$ is a constant.

Solution: The Fourier transform of the Gaussian can be found in a table:

$$\hat{f}(\xi) = \sqrt{a} e^{-\xi^2 a/2}.$$

By the convolution theorem for Fourier transforms,

$$\widehat{f * f} = \hat{f}^2.$$

Therefore

$$\widehat{f * f}(\xi) = a e^{-\xi^2/a} = \frac{\sqrt{a}}{\sqrt{2}} \sqrt{2a} e^{-\xi^2/a}.$$

Again from the formula sheet and the uniqueness of the Fourier transform, this time with $1/(4a)$ in place of a,

$$(f * f)(x) = \sqrt{\frac{a}{2}} e^{-x^2/(4a)}.$$

Problem 4 *TMA4130 Mathematics 4N:* [6 points]

Show that the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

can be put into the form

$$\frac{\partial^2 u}{\partial y\, \partial z} = 0,$$

via the change-of-variables $y = x + t$, and $z = x - t$.

Solution: A direct computation gives

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\frac{\partial z}{\partial x} \\
&= \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2}.
\end{aligned}$$

Likewise,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2}.$$

Taking a difference and dividing by 4 on both sides yields the desired equation.

Problem 4 *TMA4135 Mathematics 4D:* [6 points]

Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} x + 2 & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

Solution: We have
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\omega}(x+2)\,dx.$$
Notice that
$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\omega}(x+2)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\omega}x\,dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 2e^{-ix\omega}\,dx.$$
We have
$$\int_{-1}^{1} e^{-ix\omega}x\,dx = \int_{-1}^{1} \left(\frac{e^{-ix\omega}}{-i\omega}\right)' x\,dx = \frac{e^{-ix\omega}}{-i\omega}x\Big|_{-1}^{1} - \int_{-1}^{1} \left(\frac{e^{-ix\omega}}{-i\omega}\right)\,dx,$$
which gives
$$\int_{-1}^{1} e^{-ix\omega}x\,dx = \frac{e^{-i\omega}+e^{i\omega}}{-i\omega} - \frac{e^{-i\omega}-e^{i\omega}}{(-i\omega)^2} = \frac{2\cos(\omega)}{-i\omega} - \frac{2i\sin(\omega)}{\omega^2} = \frac{2i(w\cos(\omega)-\sin(\omega))}{\omega^2}.$$
We also have
$$\int_{-1}^{1} 2e^{-ix\omega}\,dx = \frac{2(e^{-i\omega}-e^{i\omega})}{-i\omega} = \frac{4\sin(\omega)}{\omega}.$$
Thus
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{2i(w\cos(\omega)-\sin(\omega))}{\omega^2} + \frac{4\sin(\omega)}{\omega}\right).$$

Problem 5 [14 points]

Find the solution to the following initial boundary value problem on $[0,\pi]$ using separation-of-variables:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t)=0, \quad u(\pi,t)=0, \quad u(x,0)=\sin(2x)+\sin(4x).$$

Solution: Postulating the ansatz $u(t,x)=F(x)G(t)$, and inserting it into the heat equation we find
$$F(x)\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}^2F}{\mathrm{d}x^2}G(t).$$
This gives us
$$\frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}x^2},$$

which is an equation with a function of t on the left and a function of an independent variable x on the right. Therefore both sides must be equal to some constant k, i.e.,

$$\frac{1}{G_k}\frac{\mathrm{d}G_k}{\mathrm{d}t} = \frac{1}{F_k}\frac{\mathrm{d}^2F_k}{\mathrm{d}x^2} = k,$$

The boundary conditions applied to the ansatz give:

$$G(t)F(0) = G(t)F(\pi) = 0.$$

The only non-trivial solution is $F(0) = F(\pi) = 0$.
If $k = 0$, we find that

$$\frac{\mathrm{d}^2F_k}{\mathrm{d}x^2} = 0 \qquad \Longrightarrow \qquad F_0(x) = a_0x + b_0.$$

This is only compatible with the boundary conditions if $a_0 = b_0 = 0$.
Likewise if $k > 0$, we find

$$\frac{\mathrm{d}^2F_k}{\mathrm{d}x^2} = kF_k, \tag{1}$$

which has the general solution

$$F_k(x) = A_ke^{\sqrt{k}t} + B_ke^{-\sqrt{k}t}.$$

Again, there are no non-zero constants A_k, B_k for which F_k can satisfy the boundary conditions.
It remains then to $k = -p^2 < 0$ for existence of non-trivial solutions. We find general solutions to (1) are

$$F_p(x) = A_p\cos(px) + B_p\sin(px),$$

where we have re-labelled the solutions F_k by F_p with $p > 0$.
The boundary conditions impose the following restrictions:

$$F_p(0) = 0 \implies A_p = 0, \qquad F_p(\pi) = 0 \implies p \in \mathbb{Z}.$$

But we also assumed (in order not to double-count) that $p > 0$. Hence $p \in \mathbb{N}\backslash\{0\}$.
Turning to the equation for G_p, we find

$$G_p(t) = C_pe^{-p^2t}.$$

Writing $D_p = B_pC_p$, we use the linearity of the PDE to write a solution of the form

$$u(t,x) = \sum_{p=1}^{\infty} D_p\sin(px)e^{-p^2t}.$$

This is enough for us to fit the initial condition. Assuming continuity in t of the series, we can write

$$f(x) = u(0,x) = \sum_{p=1}^{\infty} D_p \sin(px).$$

By extending f in an odd fashion to $[-\pi, \pi]$, to

$$\tilde{f}(x) = \begin{cases} f(x) & 0 \le x < \pi \\ -f(-x) & -\pi \le x < 0 \end{cases},$$

and then periodically over $\mathbb{R}$

$$\begin{aligned} f_-(x) &= \sum_{n\in\mathbb{Z}} \tilde{f}(x - 2n\pi)\mathbb{1}_{[(2n-1)\pi,(2n+1)\pi)} \\ &= \sum_{n\in\mathbb{Z}} f(x-2n\pi)\mathbb{1}_{[2n\pi,(2n+1)\pi)} - f(-x+2n\pi)\mathbb{1}_{[-(2n+1)\pi,-2n\pi)} \end{aligned}$$

we see that $f_-(x)$ is an odd function whose restriction to $[0, \pi)$ is f.
Therefore using the inversion formula, we can write

$$D_n = \frac{1}{2\pi}\int_0^{\pi} f_-(x)\sin(nx)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

This determines a solution $u(t, x)$ completely.

Problem 6 [8 points]

Use the central difference method to discretize the second-order equation:

$$u'' + 4xu = r(x), \qquad x \in [2,5], \qquad u(2) = 3, \quad u(5) = 4.$$

That is, with an appropriate discretization of $[2, 5]$ into intervals of length h, using the approximation $U_i \approx u(x_i)$, and writing $R_i = r(x_i)$, write down the discrete approximation to the differential equation involving the second-order central difference of u at x_i.

Solution: The domain can be discretized using

$$h = \frac{5-2}{N}, \qquad x_i = 2 + ih, \qquad i = 0, \ldots, N.$$

Discretizing the equation, we use the central difference approximation

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2),$$

so that

$$u''(x_i) \approx \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}.$$

Hence the equation can be discretized by the $N+1$ formulae:

$$U_{i+1} - 2U_i + U_{i-1} + 4x_i h^2 U_i = R_i h^2, \qquad n = 1, \dots, N-1,$$

and

$$U_0 = 3, \quad U_N = 4.$$

Problem 7 [8 points]

a) Given the ordinary differential equation

$$y' = x^2 y, \quad y(0) = 1.$$

Write down the implicit (backward) Euler method for this equation for a given step size h.

b) Choose $h = 0.1$ and compute an approximate value for $y(0.2)$

Solution:

(a) First we discretize the half-line : $\mathbf{x}_n = nh$. The implicit Euler scheme is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \cdot f(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}).$$

Putting in $f(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}) = \mathbf{x}_{n+1}^2 \mathbf{y}_{n+1} = (n+1)^2 h^2 \mathbf{y}_{n+1}$, we arrive at

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{x}_{n+1}^2 \mathbf{y}_{n+1} = \mathbf{y}_n + (n+1)^2 h^3 \mathbf{y}_{n+1},$$

with initial condition $\mathbf{y}_0 = 1$.

(b) Solving for $\mathbf{y}_{n+1}$ at each step, one finds:

$$(1 - (n+1)^2 h^3)\mathbf{y}_{n+1} = \mathbf{y}_n.$$

First, from the discretization $\mathbf{x}_n = nh$, we can assume that $y(0.2) \approx \mathbf{y}_2$. Therefore calculating as follows:

$$\mathbf{y}_1 = \frac{1}{(1-h^3)}\mathbf{y}_0 = \frac{1}{0.999},$$
$$\mathbf{y}_2 = \frac{1}{(1-2^2h^3)}\mathbf{y}_1 = \frac{1}{0.996} \cdot \frac{1}{0.999},$$

we can conclude that

$$y(0.2) \approx 1.005021085,$$

keeping in mind that the global error is $O(h)$.

Problem 8 [7 points]

Find the interpolation polynomial of lowest degree for the following points:

x_n	-4	0	1	2
$f(x_n)$	-85	-9	0	29

Solution:
The lowest degree polynomial that can be used to interpolate three points is cubic. The solution is $2x^3 + 4x^2 + 3x - 9$.

Problem 9 [10 points]

Recall the following difference formula for a four times continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$:

$$f''(a) \text{ can be approximated by } \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Assume that $|f^{(4)}(x)| \leq 1$ for all $x \in \mathbb{R}$. Use fourth order Taylor expansion to show the following error estimate

$$\left| f''(a) - \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \right| \leq \frac{h^2}{12}.$$

Solution: Use Taylor's expansion.

Problem 10 [12 points]

a) Show that the equation

$$e^{-x^2} = x$$

has a unique solution on the real line.

b) Write down a bisection method for this equation, using $[0, 1]$ as the initial interval, and compute the next iteration.

c) Write down the Newton method for this equation. Compute the next iteration x_1, using $x_0 = 0.5$ as the initial point.

Solution: (a) Since the left-hand side of the equation is always positive, we cannot have any solution for x non-positive. Consider the function

$$f\colon [0,\infty) \to \mathbb{R}, \quad f(x) = e^{-x^2} - x$$

We have

$$f'(x) = -2xe^{-x^2} - 1 = -\left(2xe^{-x^2} + 1\right) < 0$$

which shows that f is strictly decreasing. Since $f(0) = 1 > 0$ and $f(1) = e^{-1} - 1 < 0$ the function f has a unique zero on the interval $[0,1]$ and no zeros outside this interval.

(b) [Method p. 2 in notes] Let $a_0 = 0$ and $b_0 = 1$ and $c_0 = (a_0 + b_0)/2 = 0.5$. Since $f(c_0) = e^{-0.5^2} - 0.5 = 0.28 > 0$, we let $a_1 = c_0 = 0.28$ and $b_1 = b_0 = 1$. We find then $c_1 = (a_1 + b_1)/2 = 0.64$.

(c) The Newton iteration equals

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots.$$

We have

$$f'(x) = -2xe^{-x^2} - 1 = -\left(2xe^{-x^2} + 1\right),$$

and thus

$$x_0 = 0.5, \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} \approx 0.656735.$$