Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Mathematics 4D
Fall 2020

Exercise set 8

Submission deadline: 20th October

For the exercises (marked with J) in which you are supposed to use/modify code in the Jupyter notebook, it is enough to write down (or screen-dump) the final output.

Mandatory problems

$\boxed{1}$ (The ideas can be found in the Class Notes for Oct 13th) We consider the equation

$$f(x) = x^3 - x^2 - x + 1 = 0,$$

which has the two roots $r = -1$ and $r = +1$. In this exercise, we will discuss the usage of the Newton method in order to solve this equation. One can show that Newton's method for the solution of this equation converges for each initialisation $x_0 > 1$ to the root $r = 1$, and for each intitialisation $x_0 < -1$ to the root $r = -1$.

a) Write down the iteration scheme and perform the first three iterations with initialisation $x_0 = 2$.

b) Based upon the theory developed in the lecture, what convergence order do you expect for the iterations starting with $x_0 = 2$ and $x_0 = -2$, respectively?

c) (J) Use the code from *Preliminary.ipynb: Numerical verification of the order* to verify the convergence orders numerically.

$\boxed{2}$ Consider the data points

x_i	-2	-1	1	2
$f(x_i)$	-7	0	2	9

a) Use Lagrange interpolatation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form

$$p_n(x) = a_n x^n + \cdots + a_1 x + a_0.$$

b) Use your result to find an approximation to $f(0)$.

$\boxed{3}$ **a)** Given the points

x_i	-1	0	1
y_i	2	0	0

Set up the table of divided differences, and write down the second order interpolation polynomial in the Newton form.

b) Find the interpolation polynomial interpolating the points from **a)** and one extra point, $x_3 = 2$ and $y_3 = -4$.

Additional exercises

These additional exercises are completely optional and should not be handed in. The student assistants will not grade these problems.

$\boxed{4}$ We shall derive the Chebyshev polynomials and their zeros

a) Defining $\kappa := \cos^{-1}(x)$ for $x \in [-1, 1]$, show that $T_n(x) = \cos(n\kappa)$ and $U_n(x) = \sin(n\kappa)$ solve the Chebychev differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

for $n = 0, 1, 2, 3, 4, \ldots$. The $T_n(x)$ and $U_n(x)$ are called Chebyshev polynomials of the 1st and 2nd kind of degree n, respectively.

b) Show that $T_n(x) = \cos(n\kappa)$ satisfies the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

That is, the Chebyshev polynomials $T_0(x)$ and $T_1(x)$ imply all other polynomials by means of this recurrence formula. (Hint: use the trigonometric identities on $\cos((n \pm 1)\kappa)$.)

c) Write down the first 4 Chebyshev polynomials as polynomials in x.

d) Derive the formula for the roots of $T_{n+1}(x)$. Consider the zeros $\{x_i^{\text{cheb}}\}_{i=0}^{n}$ for $n = 3, 4, 5$ and deduce the corresponding points on the unit circle in the upper half plane. Graph these three cases.

e) Show that for Chebyshev polynomials written on the form

$$T_{n+1}(x) = c_{n+1}x^{n+1} + c_nx^n + \cdots + c_1x + c_0$$

the leading coefficient satisfies $c_{n+1} = 2^n$. Use this to prove that the polynomial

$$\omega_{\text{cheb}}(x) = \prod_{i=0}^{n}(x - x_i^{\text{cheb}})$$

satisfies

$$|\omega_{\text{cheb}}(x)| \leq \frac{1}{2^n} \text{ for all } x \in [-1, 1].$$

f) Compute the integral

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1 - x^2}}dx.$$

One says that the Chebyshev polynomials form an orthogonal set on the interval $[-1, 1]$ with a so-called weight function $1/\sqrt{1 - x^2}$ (Hint: recall the computation of the integral $\int_0^\pi \cos(n\lambda)\cos(m\lambda)d\lambda$ for integers m, n.)