Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Mathematics 4D
Fall 2020

Exercise set 7

For those exercises (marked with J) in which you are supposed to use/modify code in the Jupyter notebook, it is enough to write down (or screen-dump) the final output.

Mandatory problems

$\boxed{1}$ We consider the equation

$$f(x) = x - e^{-x} = 0.$$

a) Show that this equation has a unique solution $r \in \mathbb{R}$. Show in addition that the solution satisfies the estimate $1/2 \le x \le 1$.

b) (J) In order to compute the solution of this equation numerically, we consider the fixed point iterations

$$x_{k+1} = g(x_k),$$

where g is any of the following functions:

1. $g(x) = -\ln(x)$.
2. $g(x) = e^{-x}$.
3. $g(x) = (x + e^{-x})/2$.

Test the three iterations by using (and modifying appropriately) the function `fixedpoint` from the Jupyter notes. Use $x_0 = 1/2$ as an initial value.

Which of the iterations converges? Which of the iterations shows the fastest convergence speed?

c) Use the fixed point theorem in order to obtain a theoretical explanation for the numerical results.

$\boxed{2}$ We want to apply fixed point iteration to the solution of the equation $\cos(x) = \frac{1}{2}\sin(x)$ using the mapping

$$g(x) = x + \cos(x) - \frac{1}{2}\sin(x)$$

with an initialisation $x_0 = 0$.

a) Show that the mapping g satisfies the conditions for the fixed point theorem on the interval $[a, b] = [0, \pi/2]$.

b) Provide an estimate of the accuracy of the outcome of the method after the 5^{th} iteration. How many iterations will be needed to obtain a result with an error smaller than 10^{-12}?

Hint: Use the a-priori error estimate in the fixed point theorem.

$\boxed{3}$ **a)** Derive the formula for Newton's method for the solution of the equation $x^n = a$ where n is a natural number and $a > 0$. [1]

b) Use Newton's method to approximate a solution of the equation $x^4 = 4$. Use $x_0 = 1$ as starting value and perform four iterations by hand.

Additional exercises

These additional exercises are completely optional and should not be handed in. The student assistants will not grade these problems.

$\boxed{4}$ **a)** Let $g(x)$ be a continuous function with continuous derivatives on (a, b) and assume that it has an inverse $g^{-1}(x)$. Show that if $r \in (a, b)$ is a fixed-point of $g(x)$, then r is also a fixed-point of $g^{-1}(x)$.

b) Let $r \in (a, b)$ be a fixed-point of $g(x)$. Show that if $|g'(r)| > 1$, then $|(g^{-1})'(r)| < 1$. Using the convergence theorem of fixed-point iteration, we now know that if the algorithm does not converge for $g(x)$, then it will converge for $g^{-1}(x)$ if one initialises with x_0 sufficiently close to r.

c) With this in mind, use fixed-point iteration to find an approximation to the solution r of the equation $x = \arccos(x)$.

[1] For $n = 2$ (that is, computation of square roots), this method actually goes back to antiquity and is known as "Heron's method" or the "Babylonian method". See `http://en.wikipedia.org/wiki/Methods_of_computing_square_roots#Babylonian_method`.