TMA4135
Mathematics 4D
Fall 2020

Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 12

Submission deadline: 20th November

For the exercises (marked with J) in which you are supposed to use/modify code in the Jupyter notebook, it is enough to write down (or screen-dump) the final output.

Mandatory problems

1 Consider the Runge–Kutta method given by the following Butcher-tableau:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

a) Using the order conditions to determine the order of this method.

b) Find the stability function for this method, and show that it is not $A(0)$-stable in the complex sense.

2 Recall that the trapezoidal rule is the implicit method given by

$$y_{n+1} = y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\big).$$

Find the stability function for this method, and show that it is $A(0)$-stable in the complex sense.

Additional exercises

These additional exercises are completely optional and should not be handed in. The student assistants will not grade these problems.

3 *See the solution for 2019 Ex11, 4a)* Prove that the implicit midpoint rule

$$\vec{y}_{n+1} = \vec{y}_n + h\vec{f}\Big(x_n + \frac{h}{2}, \frac{1}{2}\vec{y}_n + \frac{1}{2}\vec{y}_{n+1}\Big)$$

is $A(0)$-stable.

4 *See the solution for 2019 Ex11, 3)* Consider the 3rd order Runge–Kutta method

$$\begin{aligned}
\vec{k}_1 &= f(x_n, \vec{y}_n), \\
\vec{k}_2 &= \vec{f}\Big(x_n + \frac{h}{2}, \vec{y}_n + \frac{h}{2}\vec{k}_1\Big), \\
\vec{k}_3 &= \vec{f}\Big(x_n + \frac{3h}{4}, \vec{y}_n + \frac{3h}{4}\vec{k}_2\Big), \\
\vec{y}_{n+1} &= \vec{y}_n + \frac{h}{9}\big(2\vec{k}_1 + 3\vec{k}_2 + 4\vec{k}_3\big).
\end{aligned}$$

a) Write down the Butcher tableau of this method and determine its order.

b) Find the stability function $R(z)$ for this method.

c) Given the system of ODEs

$$\vec{y}' = A\vec{y} \qquad\qquad \text{with } A = \begin{pmatrix} -41 & 38 \\ 19 & -22 \end{pmatrix},$$

find the largest step-size that still results in a stable solution.

d) Implement the method and verify your results numerically. Use, for instance, $\vec{y}(0) = [1, 1]^T$ as initial value and integrate over the interval $[0, 5]$. (J)