Recall

$$(*) \quad \begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \quad \downarrow \text{integrate}$$

$$y(t) - y_0 = \int_{x_0}^{t} f(x, y(x))\, dx$$

Num. $\int_{x_0}^{t}$ $\rightsquigarrow$ Num. $y(t)$

Example:

$\int_0^1 f \sim f(0)$: Euler

$\int_0^1 f \sim \frac{1}{2}(f(0)+f(1))$ $\xrightarrow{\text{adjustment}}$ Heun

$\int_0^1 f \sim \frac{1}{6}(f(0)+4f(\frac{1}{2})+f(1))$ $\xrightarrow{\text{ad.}}$ classical RK.

General Runge-Kutta

$$k_i = f\left(x_n + c_i h,\ y_n + h \sum_{j=1}^{s} a_{ij} k_j\right) \qquad i = 1, \cdots, s$$

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$$

Coefficients : Butcher table (tableau)

$$\begin{array}{c|ccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} \quad \longleftrightarrow \quad \begin{array}{c|c} c & A \\ \hline & b \end{array}$$

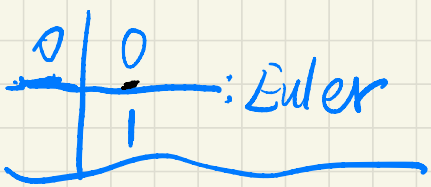
Order p: $|y_n - y(x_n)| < C \cdot n^{-p}$

Ex: ① Euler $y_{n+1} = y_n + h f(x_n, y_n)$

$\Updownarrow$

$$\begin{cases} k_1 = f(x_n, y_n) \\ y_{n+1} = y_n + h k_1 \end{cases}$$

$\Updownarrow$



: Euler

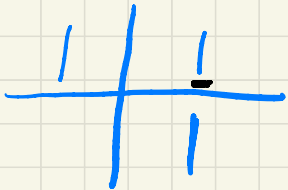
② Implicit (backward) Euler (will use it in PDE)

$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$ ← $\int_0^1 f \sim f(1)$

$\Updownarrow$

$$\begin{cases} k_1 = f(x_n + h, y_n + h k_1) \\ y_{n+1} = y_n + h k_1 \end{cases}$$

solve equ to get k_1



: Implicit Euler

Heun, "trapezoidal"

↓

Need to solve equ to get k_1

Def: A method

$$\begin{array}{c|c} C & A \\ \hline & B \end{array}$$

is explicit if $A = (a_{ij})$ satisfies that $a_{ij} = 0$ for all $j \geq i$

Ex:

$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ✓ A is strictly lower triangular

$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$ ✓

Otherwise, implicit.

RK { Euler, Heun, Classical RK : Explicit
Backward Euler, trapezoidal ; implicit }

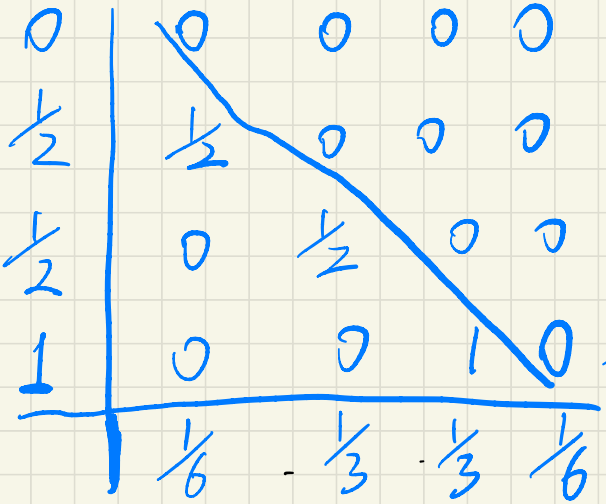
Order condition for RK (general)

Euler (order 1)

Heun (order 2)

order	conditions (NO Proof)	
1	$\sum b_i = 1$	✓
2	$\sum b_i c_i = \frac{1}{2}$	✓
3	$\sum b_i c_i^2 = \frac{1}{3}$, $\sum b_i a_{ij} c_j = \frac{1}{6}$	✓
4	$\sum b_i c_i^3 = \frac{1}{4}$, ···· Notes.	✓

Example: Classical RK



C | A
B

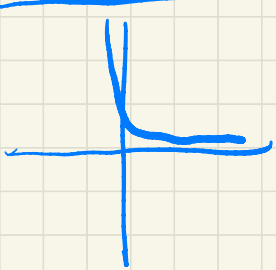
explicit

Stability analysis (for stiff ODE)

Idea:

$$\begin{cases} y' = -1000\,y \quad (*) \\ y(0) = 1 \end{cases}$$

$y = e^{-1000x}$



Euler method

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$
$$= y_n - 1000h\,y_n \quad \Leftarrow \quad f(x_n, y_n) = -1000\,y_n$$

Take $h = 0.01$ $\Rightarrow$

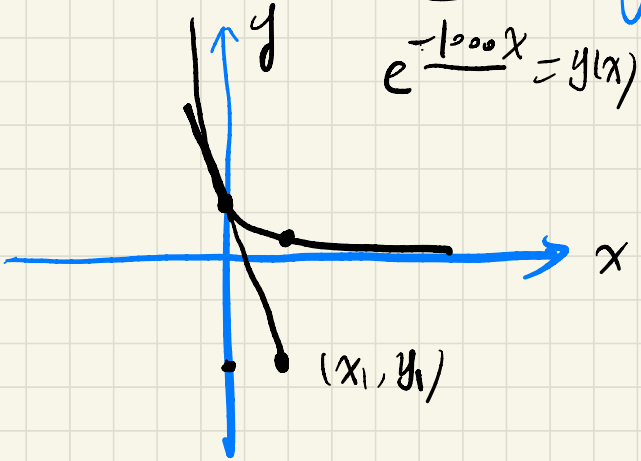
$$y_{n+1} = y_n - 10 y_n = -9 y_n$$

$$y_1 = -9 y_0 = \underline{-9}$$

$$y_2 = -9 y_1 = 81$$

$$\vdots$$

$$y_n = \underline{(-9)^n}$$ far from the exact solution!



$e^{-1000x} = y(x)$

drop down too fast!

Def: stiff ODE. $y' = \lambda y(x)$ stiff on $x > 0$

if $-\lambda$ is (large enough)

depends on specific condition.

Three notions on Stiff ODE:

for a method (Euler, Heun, RK, etc) we can study

- its stability function
- its stability interval / region
- whether it is A(0)-stable (real / complex)

Ex: (1) Euler

Step 1: Find stability function.

How?: Apply Euler to $y' = \lambda y$ $(\lambda < 0)$

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n$$

$R(z) = 1 + z$ ⟺ stability function

$$y_{n+1} = R(h\lambda)\, y_n$$

We call $R(z) := 1+z$ the stability function of Euler's method.

Step 2: Use stability function $R(z)$

Stability interval (intervall) 1-dim

$S_{\mathbb{R}} = \{ z \in \mathbb{R} : |R(z)| \leq 1 \}$

stability region (område) 2-dim

$S_{\mathbb{C}} := \{ z \in \mathbb{C} : |R(z)| \leq 1 \}$

$A(0)$-stable (Real sense) $\Leftrightarrow$

$(-\infty, 0] \subset S_{\mathbb{R}}$

$A(0)$-stable (complex sense) $\Leftrightarrow$

$\{ z \in \mathbb{C} : \mathrm{Re} \leq 0 \} \subset S_{\mathbb{C}}$

notation $\mathbb{C}^-$

For Euler: $R(z) = 1+z$

$-1 \le 1+z \le 1$

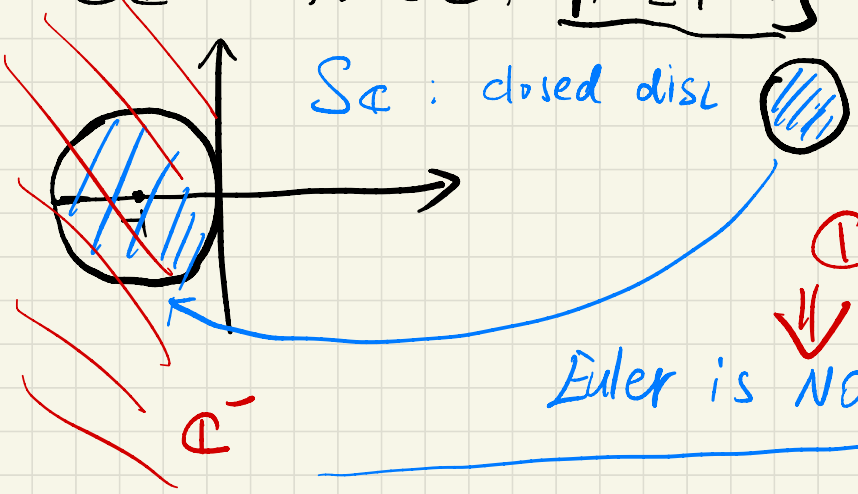
$-2 \le z \le 0$

$S_{\mathbb{R}} = \{ z \in \mathbb{R} : |1+z| \le 1 \}$

$= [-2, 0] \Rightarrow (-\infty, 0] \not\subset [-2, 0]$

$\Rightarrow$ Euler is NOT A(0)-stable (Real)

$S_{\mathbb{C}} = \{ z \in \mathbb{C} : |1+z| \le 1 \}$



$\mathbb{C}^- \not\subset S_{\mathbb{C}}$

$\Downarrow$

Euler is NOT A(0)-stable (Complex)

How to use stability interval (region) ?

$y' = \lambda y \quad (\lambda < 0)$

Euler, $y_{n+1} = R(h\lambda)\, y_n$ | $y_n = R(h\lambda)\, y_{n-1}$

$= R(h\lambda)^2\, y_{n-1}$

$= \cdots = R(h\lambda)^{n+1}\, y_0$

We hope that $|y_n|$ will NOT be divergent!

$|y_n| = |R(h\lambda)|^n y_0$ NOT divergent

$\Updownarrow$

$|R(h\lambda)| \le 1$ → motivation for stability interval

② Backward (implicit Euler)

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

$\Downarrow$ $y' = f(x,y) = \lambda y$

$$y_{n+1} = y_n + h\lambda y_{n+1}$$

$\Updownarrow$

$$(1 - h\lambda) y_{n+1} = y_n$$

$\Updownarrow$

$$y_{n+1} = \frac{1}{1-h\lambda} y_n$$

$\Downarrow$

$$R(z) = \frac{1}{1-z} \quad \left(y_{n+1} = R(h\lambda) y_n \right)$$

$\Downarrow$

$$\begin{cases} S_{\mathbb{R}} = \{ z \in \mathbb{R} \ \left|\frac{1}{1-z}\right| \le 1 \} \\ S_{\mathbb{C}} = \{ z \in \mathbb{C} : \left|\frac{1}{1-z}\right| \le 1 \} \end{cases}$$

Ex: $(-\infty, 0] \subset S_{\mathbb{R}}$; $\mathbb{C}^- \subset S_{\mathbb{C}}$

$\Rightarrow$ implicit Euler is A(0)-stable (Real / complex)!