Newton method (Theory + Usage)

Aim: Find r s.t. (such that) $f(r)=0$.

Newton method (optimal fixed point iteration)

$f(r)=0 \Leftrightarrow r=g(r)$ | g is not unique.

$\updownarrow$

$x_{k+1}=g(x_k)$

Newton: optimal $g = x - \frac{f(x)}{f'(x)}$

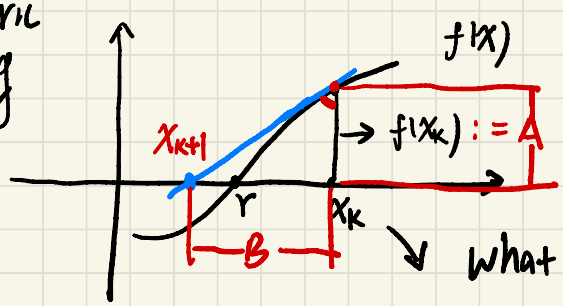
$\Downarrow$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (*)$$

equi.

Hope: $e_k := |r-x_k| \to 0 \quad (k\to\infty)$

Geometric meaning for (*):



$(*) \Leftrightarrow$

$f'(x_k) = \frac{A}{B} = \frac{f(x_k)}{x_k - x_{k+1}}$

Math. reason for (*):

$(*) \Leftrightarrow f(x_k) = f'(x_k)(x_k - x_{k+1})$ for some

$0 = f(r) = f(x_k) + f'(x_k)(r-x_k) + \frac{1}{2}f''(\xi)(r-x_k)^2$ $r \le \xi \le x_k$

$$\Rightarrow \quad 0 = f'(x_k)\,(x_k - x_{k+1} + r - x_k) + \tfrac{1}{2} f''(\xi)\,(r - x_k)^2 \qquad = r - x_{k+1}$$

$$\Updownarrow$$

$$|f'(x_k)\,(r - x_{k+1})| = \left|-\tfrac{1}{2} f''(\xi)\,(r - x_k)^2\right|$$

$$\Downarrow$$

$$|f'(x_k)| \cdot e_{k+1} = \tfrac{1}{2} |f''(\xi)| \cdot e_k^2$$

$$\Downarrow$$

$$e_{k+1} = \tfrac{1}{2} \frac{|f''(\xi)|}{|f'(x_k)|} e_k^2 \qquad (1)$$

order 2 conver.

(Assume $\le M$)

Compare with a general fixed point iteration

$e_{k+1} \le L \cdot e_k$

order 1 ↑ $0 < L < 1$

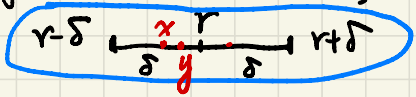
order p

$\Updownarrow$

$e_{k+1} \le A \cdot e_k^p$

Condition for order 2 conver.:

$f \in C^2[r-\delta, r+\delta]$ (f: C^2 around r)



$\left|\frac{1}{2} \frac{f''(\xi)}{f'(x)}\right| \le M$ for all $r-\delta \le x, y \le r+\delta$

$\delta \cdot M < 1$

need $f'(r) \neq 0$ key!

Error estimate ! show are enough to get order 2. C.)

Step 0: $x_0 \in [r-\delta, r+\delta]$, $L := \delta M$ $(0<L<1)$

Step 1: (1) $\Rightarrow$ $e_1 \le M \cdot e_0^2 \le M \cdot \delta^2 = L \cdot \delta$ $\le \delta$

$e_0 = |r - x_0| \le \delta$

$e_1 = |x_1 - r| < \delta$

$\Downarrow$

$x_1 \in [r-\delta, r+\delta]$ $\Downarrow$ repeat Step 0, 1:

$$e_2 \le M e_1^2 \le M(L\delta)^2 = L^2(\delta M)\cdot\delta = L^3\cdot\delta \le \delta$$

$|x_2 - r| < \delta$

$x_2 \in [r-\delta, r+\delta]$ $\Downarrow$ repeat Step 0, 1

$$e_3 \le M e_2^2 \le M\cdot(L^3\delta)^2 = L^6\cdot(\delta M)\delta = L^7\cdot\delta$$

$\vdots$ $\Downarrow$ induction

$$e_k \le L^{2^k-1}\cdot\delta \quad (0<L<1)$$ (very fast conv.)

Eks: $f(x) = x^2 - 3$ $\quad r = \sqrt{3} \approx 1.732$

Apply Newton iteration:

$x_0 = 1.75 = \frac{7}{4}$

$$x_1 = \frac{x_0^2+3}{2x_0} = \frac{\frac{49}{16}+3}{2\cdot\frac{7}{4}} = \frac{49+48}{56}$$

1.73214...

$f'(x) = 2x$

$\Downarrow$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2-3}{2x_k} = \frac{x_k^2+3}{2x_k} = \frac{2x_k^2-(x_k^2-3)}{2x_k}$$

$\delta = 0.2, \quad r = \sqrt{3}$

$1.5 < r-\delta < r+\delta < 2$

$\Downarrow \qquad x \in [1.5, 2]$

$|f'(x)| = |2x| > 3$

$\Downarrow$

$\frac{1}{2}\left|\frac{f''(y)}{f'(x)}\right| = \frac{1}{2}\cdot\frac{2}{|f'(x)|} \le \frac{1}{3} \; (:= M)$

$\Downarrow \qquad L \le M\cdot\delta = \frac{1}{3}\cdot 0.2 < 0.1$

$|x_k - r| = e_k \le (0.1)^{2^{k-1}}\cdot 0.2 \quad (*)$

$\le L^{2^{k-1}}\cdot\delta$

9-16

Break (fixed point iteration)

(1) $g([a,b]) \subset (a,b)$

(2) $|g'(x)| < L < 1 \qquad x \in [a,b]$

$\Rightarrow \exists\, 1\ r \in (a,b)$ such that $g(r) = r$

Ex 7.2: $x = e^{-x} = g(x)$

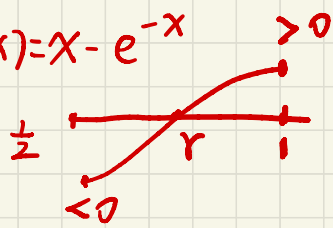
(1) X $\longleftrightarrow$ X $[\frac{1}{2}, 1]$

$1 > g(\frac{1}{2}) = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} > \frac{1}{2}$

$g(1) = e^{-1} < \frac{1}{2}$ ✗

(1) ✓ ✓ $[\frac{1}{3}, 1]$ $\quad g(1) > \frac{1}{3}$

reason: $\exists\ r \in [\frac{1}{2}, 1]$ $f(x) = x - e^{-x}$



$$|g'(r)| = |e^{-r}|$$

$$= e^{-r} < e^{-\frac{1}{2}} < 1$$

$\Downarrow$

$\exists\ \delta$ s.t. (such that) $\left(g[r-\delta, r+\delta] \subset (r-\delta, r+\delta)\right)$

$$\lim_{\Delta \to 0} \left| \frac{g(r+\Delta) - g(r)}{r+\Delta - r} \right| < 1$$

$$\lim_{\Delta \to 0} \frac{|g(r+\Delta) - r|}{|\Delta|} < 1$$

$\Downarrow$

for $|\Delta|$ small

$$|g(r+\Delta) - r| < |\Delta|$$

$\Downarrow$

$$g(r+\Delta) \in (r-|\Delta|, r+|\Delta|)$$

$\Downarrow$

key!

If $|g'(r)| < 1$ then we can choose δ :

(1) holds on $[r-\delta, r+\delta]$

(2) ✓

$e_k \le L^k e_0$ ✓

Ex 8.1 $f(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$

$\Downarrow$

f: two roots $r = 1, -1$

$r=1$ double root

$\Downarrow$

$f'(1) = 0$

(Never apply Newton method)

?

Question: How to use the Newton method ?

Newton: $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

$f'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1)$

$\Downarrow$

$x_k - \dfrac{(x_k-1)^2(x_k+1)}{(3x_k+1)(x_k-1)}$

$\Downarrow$

$\dfrac{x_k(3x_k+1) - (x_k^2-1)}{3x_k+1} = \dfrac{2x_k^2 + x_k + 1}{3x_k+1}$

$x_0 = 2$, $\Rightarrow$ 8.1 a)

$x_1 = \dfrac{2\cdot 2^2 + 2 + 1}{3\cdot 2 + 1} = \dfrac{11}{7} \simeq 1.571$

$x_2 = \dfrac{2x_1^2 + x_1 + 1}{3x_1 + 1} \sim 1.314$

$x_3 \sim 1.167$

$x_4 \sim 1.087$ (close to 1 NOT so FAST)

Why it works? Go back to iteration.

$$x_{k+1} = \frac{2x_k^2 + x_k + 1}{3x_k + 1}$$

Idea: think of it as a fixed point i.

for $g(x) = \frac{2x^2+x+1}{3x+1}$ | $x_{k+1} = g(x_k)$

key for F.P.I. is $|g'(1)| < 1$

Now $g'(x) = \frac{(4x+1)(3x+1) - 3(2x^2+x+1)}{(3x+1)^2}$

$$= \frac{6x^2+4x-2}{(3x+1)^2} \Rightarrow |g'(1)| = \frac{8}{4^2} = \frac{1}{2} < 1$$

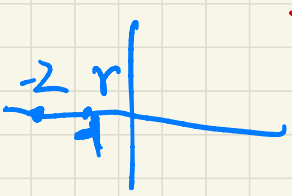
$\Downarrow$

F.P.I theorem implies that $x_{k+1} = g(x_k)$ has order 1 conv.

Ex 8.1.b): $x_0 = 2$ (order 1 conv. from FDI)

$x_0 = -2$ $\Downarrow$ $x_{k+1} = \frac{2x_k^2 + x_k + 1}{3x_k + 1}$

$x_1 = \frac{8-2+1}{-5} = \frac{7}{-5} = -1.4$



$x_2 = -1.1$, $x_3 \sim -1.0087$ ($\to -1$)

Reason: $f'(-1) = 4 \ (\neq 0)$

$\Downarrow \ \exists M$

$\frac{1}{2}\left|\frac{f''(y)}{f'(x)}\right| \leq M$ for x, y near -1

$\Downarrow$ Newton

order 2 convergence (start from $x_0 = -2$)

b)

c): Use the code from Prel.

System of non-linear equations

$$\vec{f}(\vec{x}) = 0$$

$$\Updownarrow$$

$$\begin{cases} f_1(x_1, \cdots, x_n) = 0 \\ f_n(x_1, \cdots, x_n) = 0 \end{cases}$$

Ex: $\begin{cases} x_1^3 - x_2 + \frac{1}{4} = 0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$

$\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Updownarrow \qquad \Updownarrow$$

$$\vec{x}_{k+1} = \vec{x}_k - J(\vec{x}_k)^{-1} \vec{f}(\vec{x}_k)$$

$J(\vec{x}_k)$: Jacobian matrix at $\vec{x}_k$

$$\begin{pmatrix} (f_1)_{x_1}(\vec{x}_k) & \cdots & (f_1)_{x_n}(\vec{x}_k) \\ \vdots & & \\ (f_n)_{x_1}(\vec{x}_k) & \cdots & (f_n)_{x_n}(\vec{x}_k) \end{pmatrix}$$

What is $\vec{x}_1$ (for Newton iteration)

$$\vec{x}_1 = \vec{x}_0 - J(\vec{x}_0)^{-1}\,\vec{f}(\vec{x}_0)$$

$$\vec{f}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1-1+\frac{1}{4}\\1+1-1\end{pmatrix} = \begin{pmatrix}\frac{1}{4}\\1\end{pmatrix}$$

$$J = \begin{pmatrix}3x_1^2 & -1\\ 2x_1 & 2x_2\end{pmatrix} \Rightarrow J(\vec{x}_0) = \begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}}{ad-bc} \Rightarrow J(\vec{x}_0)^{-1} = \frac{\begin{pmatrix}2 & 1\\ -2 & 3\end{pmatrix}}{8}$$

$$\vec{x}_1 = \begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{8}\begin{pmatrix}2 & 1\\ -2 & 3\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{8}\begin{pmatrix}\frac{1}{2}+1\\ -\frac{1}{2}+3\end{pmatrix}$$

$$= \begin{pmatrix}1-\frac{1}{8}\cdot\frac{3}{2}\\ 1-\frac{1}{8}\cdot\frac{5}{2}\end{pmatrix} = \dots$$