FOURIER–LAPLACE THEORY

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These notes for TMA4135 are based on

1. Erwin Kreyszig's book [3];
2. Dag Wessel-Berg's video: *http://video.adm.ntnu.no/serier/4fe2d4d3dbe03*;
3. References [2, 4, 5] (among them [2] is very short and readable).

CONTENTS

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1. LAPLACE TRANSFORM

1.1. **Basic facts.** From Wikipedia [Laplace transform]: The Laplace transform is used frequently in engineering and physics; the output of a linear time-invariant system can be calculated by convolving its unit impulse response with the input signal. Performing this calculation in Laplace space turns the convolution into a multiplication; the latter being easier to solve because of its algebraic form.... The Laplace transform can also be used to solve differential equations and is used extensively in mechanical engineering and electrical engineering. The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The original differential equation can then be solved by applying the inverse Laplace transform.

Definition 1. *Let $f(t), t \geq 0$ be a given function. We call*

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt,$$

the ***Laplace transform*** *of $f(t)$ and write $F = \mathcal{L}(f),\ f = \mathcal{L}^{-1}(F)$.*

Sometimes the Laplace transform is computable, the main idea is to use integration by parts:

$$\int_a^b f'(t)g(t)\, dt = (fg)\big|_a^b - \int_a^b f(t)g'(t)\, dt, \tag{1}$$

which follows from

$$(fg)' = f'g + g'f \quad \text{and} \quad \int_a^b (fg)'(t)\, dt = (fg)\big|_a^b. \tag{2}$$

Example 1: $f(t) \equiv 1$, write $\mathcal{L}(f)$ as $\mathcal{L}(1)$, compute that

$$\mathcal{L}(1) = 1/s, \ \text{so by our definition } \mathcal{L}^{-1}(1/s) = 1, \ \ s > 0. \tag{3}$$

Example 2: $f(t) = e^{kt}$, write $\mathcal{L}(f)$ as $\mathcal{L}(e^{kt})$

$$\mathcal{L}(e^{kt}) = \frac{1}{s-k}, \ \ s > k. \tag{4}$$

Laplace transform is linear: By linearity we mean for all numbers a, b,

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \tag{5}$$

Application 1:

$$\mathcal{L}(3 + 2e^{5t}) = 3\mathcal{L}(1) + 2\mathcal{L}(e^{5t}) = \frac{5(s-3)}{s-5}, \ s > 5.$$

Application 2: Since

$$\frac{1}{s^2-3s+2}=\frac{1}{(s-1)(s-2)}=\frac{1}{s-2}-\frac{1}{s-1},$$

linearity gives

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-3s+2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)-\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)=e^{2t}-e^t.$$

Laplace transform formula: $\mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$, $n=1,2,\cdots$.

1.2. **Laplace transform of derivatives, ODEs.** The following result leads a very quick precise solution of many ODEs.

Theorem 1 (Laplace transform of the derivative)**.**

$$\mathcal{L}(f')=s\mathcal{L}(f)-f(0).$$

Example: Apply the Laplace transform to

$$y'=y,\ y(0)=1,$$

we get

$$sY(s)-1=Y(s),$$

which gives

$$Y(s)=\frac{1}{s-1}.$$

Thus The answer is

$$y(t)=\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)=e^t.$$

$\mathcal{L}(f'')=s^2\mathcal{L}(f)-sf(0)-f'(0)$: In fact, apply the above theorem to f' one gets

$$\mathcal{L}(f'')=sL(f')-f'(0)=s(s\mathcal{L}(f)-f(0))-f'(0).$$

Example: Apply the Laplace transform to

$$y''+ay'+by=c(t),\ a,b\in\mathbb{R},$$

one gets

$$s^2Y-sy(0)-y'(0)+a(sY-y(0))+bY=C,$$

i.e

$$(s^2+as+b)Y=(s+a)y(0)+y'(0)+C,$$

thus the inverse transform gives the solution

$$y=\mathcal{L}^{-1}\left(\frac{(s+a)y(0)+y'(0)+C}{s^2+as+b}\right).$$

Application: Consider

$$y''+4y'+4y=0,\ \ y(0)=0,\ y'(0)=1,$$

the above formula gives

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2+4s+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right).$$

In order to compute the inverse Laplace transform of $\frac{1}{(s+2)^2}$, one recall that

$$\mathcal{L}(t) = \int_0^\infty te^{-st}\,dt = \frac{1}{s^2}.$$

Replace s by $s+2$, one gets

$$\int_0^\infty te^{-(s+2)t}\,dt = \frac{1}{(s+2)^2}.$$

Since $e^{-(s+2)t} = e^{-2t}e^{-st}$, the above formula gives

$$y = \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = te^{-2t}.$$

In general, replace s by $s-a$ in

$$F(s) = \int_0^\infty e^{-st}\,dt,$$

one gets

Theorem 2 (s-Shifting theorem)**.**

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Example: If we know $\mathcal{L}(1) = 1/s$ then the above formula gives

$$\mathcal{L}(e^{kt}) = \mathcal{L}(e^{kt}1) = 1/(s-k).$$

In particular, take a complex $k = iw$ one gets

$$\mathcal{L}(e^{iwt}) = \frac{1}{s-iw} = \frac{s+iw}{s^2+w^2}.$$

By Euler's formula $e^{iwt} = \cos wt + i\sin wt$, it gives (compare the real and imaginary parts)

$$\mathcal{L}(\cos wt) = \frac{s}{s^2+w^2}, \quad \mathcal{L}(\sin wt) = \frac{w}{s^2+w^2}. \tag{6}$$

Application: (6) gives

$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \cos 2t + \sin 2t.$$

1.3. **Homework 1.** Try to do the following exercises and check your result.

Exercise 1: Find the inverse Laplace transform of $\frac{s}{s^2+2s+2}$. The answer is

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+2}\right) = e^{-t}(\cos t - \sin t).$$

Exercise 2: Solve $y'' - y = t,\ y(0) = y'(0) = 1$. The answer is

$$y = e^t + \frac{1}{2}(e^t - e^{-t}) - t.$$

Definition 2 ($\sinh t$ and $\cosh t$)**.**

$$\sinh t := \frac{e^t - e^{-t}}{2}, \quad \cosh t := \frac{e^t + e^{-t}}{2}.$$

Exercise 3: Compute Laplace transform of $\sinh t$ and $\cosh t$. The answer is

$$\mathcal{L}(\sinh t) = \frac{1}{s^2-1},\ \mathcal{L}(\cosh t) = \frac{s}{s^2-1}.$$

Exercise 4: Compute Laplace transform of

$$f(t) = 1 \ \text{ if } 3 < t < 4; \quad f(t) = 0 \ \text{ otherwise}.$$

The answer is

$$\mathcal{L}(f) = \frac{e^{-3s} - e^{-4s}}{s}.$$

1.4. **Laplace transform of integrals (optional but related to the exam).** The main idea is to use the following fundamental formula in calculus

$$g(t) = \int_0^t f(\tau)\, d\tau \Rightarrow g' = f,\ g(0) = 0.$$

Thus $F = \mathcal{L}(f) = \mathcal{L}(g') = sG - g(0) = sG$ gives $G = \frac{F}{s}$, i.e.

$$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{\mathcal{L}(f)}{s}.$$

The equivalent inverse formula is useful (see the Exercise below) in inverse transforms of $F(s)/s$

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\, d\tau.$$

Exercise: Try to use it to verify the following inverse formulas

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t.$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s(s-1)}\right) = e^t - 1 - t.$$

1.5. **Using step functions.** Sometimes we have to compute Laplace transform of a piecewise smooth function, a simple and nice approach is based on the following :

Definition 3 (Step function)**.** *Let $a \geq 0$, the step function $u(t-a)$ is defined as follows*

$$u(t-a) = \begin{cases} 0 & \text{if } 0 \leq t < a; \\ 1 & \text{if } t \geq a. \end{cases}$$

In case $a = 0$ we call $u(t)$ the Heaviside function.

Examples: Draw the graphs of the following piecewise smooth functions

$$f(t) = u(t-1) - u(t-3),$$

$$f(t) = (u(t) - u(t-\pi)) \sin t,$$

$$f(t) = u(t) + u(t-1) + \cdots + u(t-n) + \cdots,$$

compare the graph of $u(t-a)f(t-a)$ with that of $f(t)$.

$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$: Notice that

$$\mathcal{L}(u(t-a)) = \int_a^\infty e^{-st}\, dt = \frac{e^{-as}}{s}. \tag{7}$$

Example: Check that

$$\mathcal{L}\left(\sum_{n=0}^{\infty} u(t-n)\right) = \frac{1}{s(1-e^{-s})}.$$

Hint: use $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

A generalization of (7) is

$$\mathcal{L}(u(t-a)f(t-a)) = \int_a^\infty e^{-st} f(t-a)\, dt.$$

Replace $t-a$ by t one gets

$$\mathcal{L}(u(t-a)f(t-a)) = \int_0^\infty e^{-s(t+a)} f(t)\, dt = e^{-as}\mathcal{L}(f).$$

Thus we have the following:

Theorem 3 (t-Shifting theorem)**.**

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f).$$

Example: Since

$$\frac{1}{s-2} = \mathcal{L}(e^{2t}),$$

we have

$$\mathcal{L}^{-1}\left(e^{-s}\frac{1}{s-2}\right) = u(t-1)e^{2(t-1)}.$$

RC-Circuit equation (optional but related to the exam, see page 29 section 1.5 and page 93 section 2.9 of Kreyszig's book)

$$R\,i(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = e(t),$$

where R, C: positive constants, $i(t), e(t)$: functions. Apply the Laplace transform we get

$$R\,I(s) + \frac{1}{C}\cdot\frac{I(s)}{s} = E(s),$$

i.e.

$$I(s) = \frac{E(s)}{R + \frac{1}{Cs}} = \frac{s}{s + \frac{1}{RC}}\frac{E(s)}{R}.$$

Example: Compute $i(t)$ when

$$e(t) = u(t-1) - u(t-2), \ \ R = C = 1.$$

Answer

$$i(s) = u(t-1)e^{-(t-1)} - u(t-2)e^{-(t-2)}.$$

1.6. **Dirac delta function.** The delta function $\delta(t-a)$ is defined by

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a).$$

In case $f(t) = e^{-st}$, we have

$$\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$$

Thus

$$\mathcal{L}(\delta(t-a)) = e^{-as}.$$

Exercise: solve

$$y'' + y = \delta(t-1), \ \ y(0) = y'(0) = 0.$$

Answer

$$y = \mathcal{L}^{-1}(e^{-s}\mathcal{L}(\sin t)) = u(t-1)\sin(t-1).$$

1.7. **Convolution.** Let $f(t), g(t)$ be two functions for $t \geq 0$.

Definition 4 (Convolution of f and g)**.**

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau, \qquad t \geq 0.$$

Examples: verify the followings:

$$1 * t = \frac{t^2}{2},$$

$$e^t * e^t = te^t,$$

$$f(t) * 1 = \int_0^t f(\tau)\,d\tau.$$

Laplace transform relates convolution to multiplication:

Theorem 4 (Laplace transform of convolution)**.**

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g).$$

1.8. **Homework 2.** Try to do the following exercises and check your result.

Exercise 1: Compute $t^m * t^n$. Hint: use $t^m * t^n = \mathcal{L}^{-1}\mathcal{L}(t^m * t^n)$, answer

$$t^m * t^n = \frac{m!n!}{(m+n+1)!}t^{m+n+1}, \ m, n = 0, 1, \cdots$$

Exercise 2: Compute $\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2})$. Hint: use $\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \mathcal{L}^{-1}(\mathcal{L}(\sin t) \cdot \mathcal{L}(\sin t))$. Answer

$$\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \frac{\sin t - t\cos t}{2}.$$

Exercise 3: Solve $y'' + y = \sin t, \ y(0) = 0, \ y'(0) = 1$. Hint: use Exercise 2. Answer

$$y = \sin t + \frac{\sin t - t\cos t}{2} = \frac{3\sin t - t\cos t}{2}.$$

Exercise 2: Solve $y - \int_0^t (t-\tau)y(\tau)\,d\tau = 1$. Hint: use $\int_0^t (t-\tau)y(\tau)\,d\tau = y * t$. Answer

$$y = \frac{e^t + e^{-t}}{2} = \cosh t.$$

1.9. **Non-homogeneous linear ODEs (optional but related to the exam).** Theorem 4 can be used to solve the following equation

$$y'' + by' + cy = r(t),$$

for given $y(0)$ and $y'(0)$. Apply the Lap. Tra., we have

$$s^2Y - sy(0) - y'(0) + b(sY - y(0)) + cY = R(s).$$

Thus

$$Y = \frac{1}{s^2 + bs + c} \cdot R(s) + \frac{sy(0) + y'(0) + by(0)}{s^2 + bs + c} := K(s) \cdot R(s) + G(s)$$

and Theorem 4 gives

$$y = k * r + g.$$

Example: Consider

$$y'' + y = r(t), \;\; y(0) = y'(0) = 0.$$

Apply the Laplace transform we have

$$s^2Y + Y = \mathcal{L}(r),$$

thus

$$Y = \frac{1}{s^2 + 1} \cdot \mathcal{L}(r)$$

and Theorem 4 gives

$$y(t) = \sin t * r.$$

1.10. **Derivative of the Laplace transform (optional but related to the exam).** Apply the differential to

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

we get

$$F'(s) = \int_0^\infty \frac{d(e^{-st})}{ds} f(t)\, dt = \int_0^\infty e^{-st} \cdot (-tf(t))\, dt = \mathcal{L}(-tf(t)).$$

Example 1: Since $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$, the above formula gives

$$\mathcal{L}(t \sin t) = -\left(\frac{1}{s^2 + 1}\right)' = \frac{2s}{(s^2 + 1)^2}.$$

Example 2: Let $F(s) = \ln(1 + s^{-2})$. Then

$$F' = (\ln(1 + s^2) - \ln(s^2))' = \frac{2s}{1 + s^2} - \frac{2}{s}.$$

Thus

$$\mathcal{L}^{-1}(F') = 2\cos t - 2.$$

By the above theorem, we have $\mathcal{L}^{-1}(F') = -tf(t)$, thus

$$f(t) = \mathcal{L}^{-1}(\ln(1 + s^{-2})) = \frac{2 - 2\cos t}{t}.$$

1.11. **Homework 3.** Try to do the following exercises and check your result.

Exercise 1: Compute Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le a; \\ 0 & \text{if } t > a. \end{cases}$$

Answer:

$$F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}.$$

Exercise 2: Compute Laplace transform of $f(t) = u(t-\pi)\sin t$. Answer $F = -\frac{e^{-\pi s}}{s^2+1}$.

Exercise 3: Solve $i(t)$:

$$i'(t) + 2\,i(t) + \int_0^t i(\tau)\,d\tau = \delta(t-1),\ i(0) = 0.$$

Answer:

$$i(t) = u(t-1)(e^{-(t-1)} - e^{-(t-1)}(t-1)).$$

2. Fourier analysis

Perhaps the most important use of the Fourier analysis is to solve partial differential equations. Many of the equations of the mathematical physics of the nineteenth century can be treated this way (from Wikipedia [Fourier transform]).

J. Fourier, 1808-9: "Regarding the researches of d'Alembert and Euler could one not add that if they know this expansion, they made but a very imperfect use of it. They were both persuaded that an arbitrary and discontinuous function could never be resolved in series of this kind, and it does not even seem that anyone had developed a constant in cosines of multiple arcs, the first problem which I had to solve in the theory of heat."

2.1. **Complex Fourier series.** The main result of Fourier series is that (every smooth 2π periodic function can be expressed by simple 2π periodic functions e^{inx}). More precisely we have the following result:

Theorem 5 (Fourier 1807 [4]). *If f has period 2π and is smooth enough then we have the following complex Fourier series identity*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n\, e^{inx}, \quad c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

and the following Parseval identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |c_n|^2,$$

where c_n are the complex Fourier coefficients.

Period: Recall that f has period 2π means

$$f(x+2\pi) \equiv f(x).$$

It is clear that

$$e^{inx} = \cos nx + i \sin nx, \quad n \in \mathbb{Z},$$

are 2π periodic.

Smooth enough: It means that f is piecewise smooth and

$$f(x_0) = \frac{f(x_0+) + f(x_0-)}{2},$$

if f is not smooth at x_0, where $f(x_0+), f(x_0-)$ denote the right and left limits.

Fourier series as an orthogonal decomposition: Consider the following L^2-inner product

$$(f,g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx,$$

our theorem gives

$$c_n = (f, e^{inx}).$$

We claim the followings (which imply that $\{e^{inx}\}$ is an orthonormal system)

$$(e^{inx}, e^{imx}) = \begin{cases} 0 & n \neq m; \\ 1 & n = m. \end{cases}$$

In fact, notice that

$$(e^{inx}, e^{imx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \begin{cases} \frac{1}{2\pi i(n-m)}\left(e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right) = 0 & n \neq m; \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = 1 & n = m, \end{cases}$$

our claim follows from $e^{i(n-m)\pi} = e^{-i(n-m)\pi}$. Thus we know

$$f = \sum_{n\in\mathbb{Z}} c_n\, e^{inx}$$

is an orthogonal decomposition.

Example: Consider a piecewise smooth 2π periodic function defined by

$$f(x) = \begin{cases} 1 & 0 < x < \pi; \\ -1 & -\pi < x < 0; \\ 0 & x = 0, \pi, -\pi. \end{cases}$$

Its complex Fourier coefficients are given by

$$c_n = (f, e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi}\left(\int_0^{\pi} e^{-inx}\, dx - \int_{-\pi}^{0} e^{-inx}\, dx\right).$$

Compute

$$\int_0^{\pi} e^{-inx}\, dx = \frac{e^{-inx}}{-in}\Big|_0^{\pi} = \frac{(-1)^n - 1}{-in}, \quad \int_{-\pi}^{0} e^{-inx}\, dx = \frac{1-(-1)^n}{-in},$$

we have

$$c_n = \frac{1-(-1)^n}{\pi i n} = \begin{cases} \frac{2}{in\pi} & n\ odd; \\ 0 & n\ even. \end{cases}$$

Thus the complex fourier series of f is

$$f(x) = \sum_{m\in\mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.$$

Notice that $(f, f) = 1$, the Parseval identity gives

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

2.2. **(Real) Fourier series.** In the previous example, we have

$$f(x) = \frac{2}{i\pi}\left(e^{ix} + \frac{e^{3ix}}{3} + \cdots\right) + \frac{2}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \cdots\right),$$

thus

$$f(x) = \frac{2}{i\pi}\left((e^{ix} - e^{-ix}) + \frac{e^{3ix} - e^{-3ix}}{3} + \cdots\right).$$

Euler's formula $e^{\pm inx} = \cos nx \pm i \sin nx$ gives

$$e^{inx} - e^{-inx} = 2i \sin nx,$$

thus the complex Fourier series of f reduces to a sine series

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \cdots\right),$$

For a *general* function f, by the Euler formula, we have

$$f(x) = \sum c_n e^{inx} = \sum c_n(\cos nx + i \sin nx),$$

which gives

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n(\cos nx + i \sin nx) + \sum_{n=1}^{\infty} c_{-n}(\cos nx - i \sin nx).$$

Thus we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left((c_n + c_{-n}) \cos nx + i(c_n - c_{-n}) \sin nx\right),$$

Recall that

$$(c_n + c_{-n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx,$$

thus we get the following general result

Corollary 1. *If f has period 2π and is smooth enough then it has the following **Fourier series** expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

*where a_0, a_n, b_n are the **Fourier coefficients** of f such that*

$$a_0 = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

and for $n = 1, 2 \cdots$, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Example: Consider

$$f(x) = \begin{cases} 0 & -\pi < x < 0; \\ x & 0 \le x < \pi. \end{cases}$$

Then

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)\, dx = \int_0^{\pi} x\, dx = \frac{\pi^2}{2}$$

and an integration by parts (try) gives

$$\pi a_n = \frac{(-1)^n - 1}{n^2}, \quad \pi b_n = \frac{\pi(-1)^{n+1}}{n}, \ n = 1, 2, \cdots$$

i.e.

$$a_0 = \frac{\pi}{4}, \quad a_{2m} = 0, \ a_{2m-1} = \frac{-2}{(2m-1)^2 \pi}, \quad m = 1, 2, \cdots; \ \ b_n = \frac{(-1)^{n+1}}{n}.$$

Thus the Fourier series of f is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).$$

Take $x = 0$ we get

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \cdots\right),$$

i.e.

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Assume that

$$A := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

the above identity gives

$$A = \frac{\pi^2}{8} + \frac{1}{2^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{8} + \frac{A}{4},$$

which implies

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

2.3. **Homework 4.** Is the Fourier series expansion in Corollary 1 an orthogonal decomposition ? If so then try to find the associated Parseval identity.

2.4. **Fourier Sine and Cosine series.** Since $\sin nx$ is odd and $\cos nx$ is even, one may guess that an odd (even) function will only have sine (cosine) series.

Definition 5. *We say that f is odd if $f(-x) = -f(x)$; f is even if $f(-x) = f(x)$.*

Basic Fact: We have

$$\int_{-\pi}^{\pi} f(x)\,dx = \begin{cases} 2\int_0^\pi f(x)\,dx, & f \text{ even;} \\ 0, & f \text{ odd.} \end{cases}$$

In particular, if f is odd then $f(x)\cos nx$ are odd, thus all

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0;$$

if f is even then $f(x)\sin nx$ are odd, thus all

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

Moreover, the above Basic Fact gives:

Theorem 6. *Assume that f has period 2π and is smooth enough. If f is **odd** then it can be written as a **Fourier sine series***

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.$$

*If f is **even** then it can be written as a **Fourier cosine series***

$$f(x) = \frac{1}{\pi}\int_0^\pi f(x)\,dx + \sum_{n=1}^{\infty} a_n \cos nx, \;\; a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

One way to get odd (even) functions is to use the odd or even extension: Let f be a function in $(0,\pi)$. Then we can extend f to an odd function f_o such that

$$f_o(-x) = -f(x), \quad \forall \; x \in (0,\pi);$$

we can also extend f to an even function f_e such that

$$f_e(-x) = f(x), \quad \forall \; x \in (0,\pi).$$

2.5. **Homework 5.** Draw the graph of the odd extension f_o and the even extension f_e of

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2}; \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Find the Fourier cosine series of f_e. Answer

$$f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left(-\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots\right).$$

Find the Fourier sine series of f_o. Answer

$$\begin{aligned} f_o(x) &= \left(\frac{2}{\pi}+1\right)\sin x + \left(0-\frac{1}{2}\right)\sin 2x + \left(\frac{-2}{3^2\pi}+\frac{1}{3}\right)\sin 3x \\ &\quad + \left(0-\frac{1}{4}\right)\sin 4x + \left(\frac{2}{5^2\pi}+\frac{1}{5}\right)\sin 5x + \cdots . \end{aligned}$$

2.6. **Fourier transform.** The integral version of "Fourier analysis" is based on the Fourier transform, which has many deep and interesting applications, among them (see [6]) are the *Heisenberg Uncertainty Principle* and the *Central Limit Theorem*.

Definition 6. *We call*

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx,$$

*the **Fourier transform** of f and write $\hat{f} = \mathcal{F}(f)$.*

The main result on Fourier transform is the following:

Theorem 7. *Let f be a smooth and rapidly decreasing function. Then we have*

$$\widehat{f'} = iw\hat{f}, \quad (\hat{f})' = -\widehat{ixf}$$

*the **Fourier inversion formula***

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw}\, dw$$

*and the **Parseval-Plancherel identity***

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw.$$

Smooth and rapidly decreasing means

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \quad \text{for every } k, l \geq 0,$$

where $f^{(l)}$ denotes the l-th derivative of f. One may check that e^{-x^2} is Smooth and rapidly decreasing. Sometimes one could also compute Fourier transform of a non-smooth function (below are examples).

Example: F1: Fourier transform of

$$f(x)=\begin{cases}1, & |x|<1;\\ 0, & |x|\geq 1.\end{cases}$$

By our definition

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-iwx}\,dx$$

one computes that

$$\hat{f}(w)=\begin{cases}\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}, & w\neq 0;\\ \sqrt{\frac{2}{\pi}}, & w=0.\end{cases}$$

Example: F2: Fourier transform of

$$f(x)=\begin{cases}e^{-x}, & x>0;\\ 0, & x\leq 0.\end{cases}$$

We have

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-x}e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\mathcal{L}(e^{-iwt})(1).$$

Recall that

$$L(e^{-iwt})(s)=\frac{1}{s+iw},$$

thus

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+iw}.$$

2.7. **A canonical fixed point of the Fourier transform.** In this section we shall show that $e^{-\frac{x^2}{2}}$ is a fixed point of the Fourier transform. More precisely, we shall prove that

Proposition 1. *Let* $f(x)=e^{-\frac{x^2}{2}}$. *Then*

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}e^{-iwx}dx=e^{-\frac{w^2}{2}}=f(w). \tag{8}$$

Proof. Step 1: Look at the derivative of $\hat{f}(w)$:

$$(\hat{f})'(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}(-ix)e^{-iwx}\,dx.$$

Notice that $(e^{-\frac{x^2}{2}})'=e^{-\frac{x^2}{2}}(-x)$, thus

$$(\hat{f})'(w)=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{-\frac{x^2}{2}})'e^{-iwx}\,dx=\frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}(e^{-iwx})'\,dx=-w\hat{f}(w),$$

which gives

$$\left(\hat{f}(w)e^{\frac{w^2}{2}}\right)' = \left((\hat{f})'(w) + w\hat{f}(w)\right) e^{\frac{w^2}{2}} = 0.$$

Thus $\hat{f}(w)e^{\frac{w^2}{2}}$ is a constant, i.e.

$$\hat{f}(w)e^{\frac{w^2}{2}} \equiv \hat{f}(0)e^0 = \hat{f}(0),$$

which gives

$$\hat{f}(w) = \hat{f}(0)e^{\frac{-w^2}{2}} = \hat{f}(0)f(w).$$

Step 2: Since $\hat{f}(w) = \hat{f}(0)f(w)$ and $f(-x) = f(x)$, by the Fourier inversion formula we have

$$f(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(0)f(w)e^{iwx}\, dx = \hat{f}(0)\hat{f}(-x) = (\hat{f}(0))^2 f(x),$$

which gives $\hat{f}(0) = \pm 1$. But notice that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx > 0,$$

we must have $\hat{f}(0) = 1$. Thus $\hat{f} = f$. □

In "Step 2" we have proved that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = 1,$$

which is the classical formula in Gauss's normal distribution theory, since

$$f(u\,|\,\mu, t) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(u-\mu)^2}{2t}}, \tag{9}$$

is the *probability density of the normal distribution with expectation* μ *and variance* t.

2.8. **Inverse Laplace transform (optional).** The Fourier inversion formula can be used to find the inverse Laplace transform. Recall the definition of the Laplace transform

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

Extend f to a function on $\mathbb{R}$ such that

$$f(t) = 0, \quad \forall\, t < 0.$$

Thus we have

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt,$$

which gives

$$F(iw) = \sqrt{2\pi}\hat{f}(w).$$

Thus the Fourier inversion formula gives

$$\text{Laplace inversion formula}: \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{itw}\, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(iw)e^{itw}\, dw.$$

2.9. **Fourier transform of a convolution.** Let us first recall the definition of convolution for functions f, g defined on $[0, \infty)$:

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

Notice that if we extend f, g to functions on $\mathbb{R}$ such that

$$f = g = 0, \text{ when } x \leq 0,$$

then we have

$$(f * g)(x) := \int_{-\infty}^{\infty} f(u)g(x-u)\, du,$$

which suggest the following definition:

Definition 7. *The convolution of two functions on $\mathbb{R}$ is defined by*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(u)g(x-u)\, du.$$

Similar as the Laplace transform, we have

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{10}$$

Application: The above formula can be used to solve convolution equations, in fact the Fourier transform of

$$e^{-\frac{x^2}{2}} * f(x) = g(x)$$

becomes (here we use the fact that "$e^{-\frac{x^2}{2}}$" is a fixed point of F. T.)

$$\sqrt{2\pi} \cdot e^{-\frac{w^2}{2}} \cdot \hat{f}(w) = \hat{g}(w),$$

thus

$$\hat{f}(w) = e^{\frac{w^2}{2}} \cdot \hat{g}(w)/\sqrt{2\pi}$$

and the Fourier inversion formula gives

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{\frac{w^2}{2}} \hat{g}(w)/\sqrt{2\pi}) \cdot e^{ixw}\, dw.$$

In particular if $g \equiv 0$ hence $\hat{g} \equiv 0$ then we must have $f \equiv 0$.

3. Partial differential equations

We shall show how to use Fourier analysis to solve the heat and wave equations.

3.1. **Functions of several variables.** In Laplace transform theory, the function (or map)

$$(s,t) \mapsto e^{-st},$$

is well defined on $\mathbb{R}^2$. It is clear that

$$f(x,y) = x^2 + y^2,$$

is a function on $\mathbb{R}^2$;

$$f(x,y,z) = x + y + z,$$

is a function on $\mathbb{R}^3$ and

$$f(t,x) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

is a function on $(0,\infty) \times \mathbb{R}$.

3.2. **Partial derivatives.** x-partial derivative of $f(x,y)$ means derivative of f with y fixed:

$$\frac{\partial f}{\partial x}(x,y) := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

we write

$$f_x := \frac{\partial f}{\partial x}, \quad f_{xy} := \frac{\partial f_x}{\partial y}.$$

Example 1: If $f(x,y) = x^2 + y^2 + xy$ then

$$f_x = 2x + y, \quad f_y = 2y + x, \; f_{xy} = 1 = f_{yx}, \; f_{xx} = 2, \; f_{yy} = 2.$$

Example 2: Consider the heat kernel

$$f(t,x) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

then

$$f_t = \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{2} \cdot t^{\frac{-3}{2}} \cdot e^{-\frac{x^2}{2t}} + \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}} \cdot \frac{-x^2}{2} \cdot \frac{-1}{t^2} = \frac{x^2 - t}{2t^2} f,$$

and

$$f_x = -\frac{x}{t} f, \; f_{xx} = -\frac{1}{t} f + \frac{x^2}{t^2} f = \frac{x^2 - t}{t^2} f.$$

Thus f satisfies the following heat equation

$$f_t = \frac{1}{2} f_{xx}. \tag{11}$$

Gradient of $f(x,y)$ is defined by

$$\nabla f := (f_x, f_y).$$

3.3. **Fourier series solution of wave equation.** Let us solve the wave equation

$$u_{tt} = u_{xx},$$

with boundary conditions

$$u(t, 0) = u(t, \pi) = 0, \quad \forall\, t \geq 0;$$

and initial conditions

$$u(0, x) = f(x), \; u_t(0, x) = g(x), \quad \forall\, 0 \leq x \leq \pi.$$

Step 1: Separate variables: Find solutions of the form

$$u(t, x) = G(t)F(x),$$

Since

$$u_{tt} = G''F, \;\; u_{xx} = GF'',$$

our equation becomes

$$G''F = GF'',$$

thus

$$\frac{G''}{G} = \frac{F''}{F} \equiv k,$$

where k is constant (notice that k does not depend on t and x).

Step 2: Fit boundary conditions: Notice that the boundary conditions

$$G(t)F(0) = G(t)F(\pi) = 0,$$

is equivalent to

$$F(0) = F(\pi) = 0.$$

In case $k = 0$, then $F'' \equiv 0$, i.e $F(x) = ax + b$. The boundary conditions give $F \equiv 0$.
In case $k = \mu^2 > 0$, then the general solution for

$$F'' = \mu^2 F,$$

is $F = Ae^{\mu x} + Be^{-\mu x}$, then the boundary conditions give

$$A + B = 0, \;\; Ae^{\mu\pi} + Be^{-\mu\pi} = 0,$$

thus $A = B = 0$.

Thus the only possible case is $k = -p^2 < 0$, then the general solution for

$$F'' = -p^2 F,$$

is $F = A\cos px + B\sin px$, $F(0) = 0$ gives $A = 0$. Thus $F = B\sin px$, $B \neq 0$, but $F(\pi) = 0$ gives $\sin p\pi = 0$, i.e.

$$p = n, \;\; n = 1, 2, \cdots,$$

(notice that $\sin -px = -\sin px$, thus up to a constant they give the same solution).

Summary: The boundary condition implies that $p = n^2$, $n = 1, 2, \cdots$, and

$$F = F_n(x) = \sin nx.$$

Now let us solve

$$G'' = -n^2 G,$$

the general solution is

$$G_n(t) = B_n \cos nt + C_n \sin nt.$$

Now we know that each

$$u_n(t,x) = G_n(t)F_n(t) = (B_n \cos nt + C_n \sin nt)\sin nx,$$

satisfies the wave equation with boundary conditions, so does

$$u(t,x) = \sum_{n=1}^{\infty}(B_n \cos nt + C_n \sin nt)\sin nx.$$

Step 3: Fit the initial conditions: Choose B_n and C_n such that

$$u(0,x) = f(x), \;\; u_t(0,x) = g(x).$$

i.e.

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x);$$

and

$$\sum_{n=1}^{\infty} nC_n \sin nx = g(x);$$

Consider odd extension f_o, g_o of f, g, by Theorem 6, if f_o and g_o is smooth enough, then it is enough to choose

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx,$$

and

$$nC_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\,dx.$$

Now we know that if f, g have smooth enough odd extension then

$$u(t,x) = \sum_{n=1}^{\infty}(B_n \cos nt + C_n \sin nt)\sin nx,$$

solves the wave equation (of course we have to check that it converges).

Example: When $g = 0$ we have

$$u(t,x) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx.$$

Thus we can write

$$u(t,x) = \frac{1}{2}\sum_{n=1}^{\infty}(B_n \sin n(x-t) + B_n \sin n(x+t)) = \frac{1}{2}(f_o(x-t) + f_o(x+t)),$$

i.e. $u(t,x)$ is the superposition of two travelings of the initial wave.

Exercise: In case

$$f(x) = x, \ \ 0 \le x \le \frac{\pi}{2}, \ \ f(x) = \pi - x, \ \ \frac{\pi}{2} < x \le \pi,$$

try to draw the graph of $u(t,x)$ for $t = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}, \pi$.

3.4. **Solve heat equation by Fourier series.** Consider the heat equation

$$u_t = \frac{1}{2} u_{xx},$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \ \ \forall\, t \ge 0;$$

and initial conditions

$$u(0,x) = f(x), \ \ \forall\, 0 \le x \le \pi.$$

Step 1: Separate variables: Find solutions of the form

$$u(t,x) = G(t)F(x),$$

Since

$$u_t = G'F, \ \ u_{xx} = GF'',$$

our equation becomes

$$G'F = \frac{1}{2} GF'',$$

thus

$$\frac{2G'}{G} = \frac{F''}{F} \equiv k,$$

Step 2: Fit the boundary conditions: Same as the wave equation, we have

$$k = -n^2, \ \ n = 1, 2 \cdots, n$$

and

$$F_n(x) = \sin nx.$$

Then

$$G' = -\frac{n^2}{2} G,$$

gives

$$G_n(t) = B_n e^{-\frac{n^2}{2} t},$$

Thus the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2}{2} t} \sin nx.$$

Step 3: Fit the initial conditions: We hope

$$u(0,x) = \sum_{n=1}^{\infty} B_n \sin nx = f(x).$$

By Theorem 6, we get

$$B_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx,$$

Example: $f(x) = 2\sin x$ then

$$B_1 = 2,\;\; B_2 = \cdots = B_n = 0.$$

Thus

$$u(t,x) = 2e^{-\frac{t^2}{2}}\sin x,$$

Notice that u goes to zero as t goes to infinity.

Exercise: Find u with

$$f(x) = x,\;\; 0 \le x \le \frac{\pi}{2},\;\; f(x) = \pi - x,\;\; \frac{\pi}{2} \le x \le \pi.$$

3.5. **Solve heat equation by Fourier transform (optional but related to the exam).** Let us still consider the heat equation

$$u_t = \frac{1}{2}u_{xx}.$$

But this time we consider the initial condition for all x in $\mathbb{R}$:

$$u(0,x) = f(x),\;\; \forall\; -\infty < x < \infty,$$

assume that f is smooth and rapidly decreasing, our aim is to find the solution u that is smooth and rapidly decreasing with respect to x.

Step 1: Reduce to ODE by Fourier transform: Consider Fourier transform of u_t with respect to the x variable

$$\mathcal{F}(u_t) = \widehat{u_t}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(t,x)e^{-ixw}\,dx.$$

Then we have

$$\mathcal{F}(u_t) = \mathcal{F}\left(\frac{1}{2}u_{xx}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}u_{xx}(t,x)e^{-ixw}\,dx.$$

Recall that if u is smooth and rapidly decreasing with respect to x then

$$\int_{-\infty}^{\infty} u_{xx}e^{-ixw}\,dx = \int_{-\infty}^{\infty} e^{-ixw}\,d(u_x) = -\int_{-\infty}^{\infty} u_x\,d(e^{-ixw}) = iw\int_{-\infty}^{\infty} u_x e^{-ixw}\,dx,$$

the same computation for u gives

$$\int_{-\infty}^{\infty} u_x e^{-ixw}\,dx = iw\int_{-\infty}^{\infty} ue^{-ixw}\,dx,$$

thus we have

$$\mathcal{F}(u_t) = \frac{-w^2}{2}\mathcal{F}(u),\;\; \mathcal{F}(u) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t,x)e^{-ixw}\,dx$$

Notice that we also have

$$\mathcal{F}(u_t) = (\mathcal{F}(u))_t.$$

Thus $\mathcal{F}(u)$ satisfies the following ODE.

$$(\mathcal{F}(u))_t = \frac{-w^2}{2}\mathcal{F}(u).$$

Step 2: Solve the ODE and fit the initial condition: The general solution is

$$\mathcal{F}(u)(t,w) = c(w)e^{\frac{-w^2}{2}t}.$$

Notice that our initial condition implies

$$\mathcal{F}(u)(0,w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixw}\,dx = \mathcal{F}(f).$$

Thus

$$c(w) = \mathcal{F}(f).$$

Now we have

$$\mathcal{F}(u)(t,w) = \mathcal{F}(f)\cdot e^{\frac{-w^2}{2}t}.$$

Step 3: Use Fourier convolution formula: Recall that (see (8))

$$e^{\frac{-u^2}{2}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}}e^{-iyu}\,dy.$$

Take

$$u = w\sqrt{t},\ y = \frac{x}{\sqrt{t}},$$

we get

$$e^{\frac{-w^2}{2}t} = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2t}}e^{-ixw}\,dx = \frac{1}{\sqrt{t}}\mathcal{F}(e^{\frac{-x^2}{2t}}).$$

Thus

$$\mathcal{F}(u)(t,w) = \mathcal{F}(f)\cdot\left(\frac{1}{\sqrt{t}}\mathcal{F}(e^{\frac{-x^2}{2t}})\right)$$

and the Fourier convolution formula gives

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}(f * e^{\frac{-x^2}{2t}}) = \int_{-\infty}^{\infty} f(p)\frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp,$$

Summary: *The solution $u(t,x)$ is given by convolution of the initial temperature distribution with the heat kernel.*

Remark 1: The solution $u(t,x)$ fits the initial condition in the sense that

$$\lim_{t\to 0+}\int_{-\infty}^{\infty} f(p)\frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp = f(x).$$

Remark 2: In the above proof we assume that u is smooth and rapidly decreasing with respect to x, this is OK since according to our final solution

$$u(t,x) = \int_{-\infty}^{\infty} f(p)\frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp,$$

u is smooth and rapidly decreasing with respect to x if f is.

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