PRELIMINARIES FOR THE FIRST PART OF MATH 4D

XU WANG

CONTENTS

1. DEFINITION OF e, π AND EULER'S FORMULA

1.1. **Where does e come from ?** Recall that: Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map (here linear map means $A(au + bv) = aA(u) + bA(v)$ for all a, b in $\mathbb{C}$ and all u, v in $\mathbb{C}^n$). We call $u \neq 0$ in $\mathbb{C}^n$ an *eigenvector* of A if

$$Au = \lambda u, \tag{1}$$

where λ is a constant in $\mathbb{C}$.

What is an eigenvector of the derivative ?

By (1), we want to find function $u : \mathbb{R} \to \mathbb{C}$ such that

$$u' = \lambda u.$$

Power series method: Assume that

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

The following lemma gives:

$$u'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \cdots .$$

Lemma 1.1. $(x^n)' = n x^{n-1}, \ n = 1, 2, \cdots .$

Proof. If $n = 1$ then

$$x'(x) = \lim_{\triangle x \to 0} \frac{(x + \triangle x) - x}{\triangle x} = 1.$$

Assume the Lemma for $n = 1, \cdots, N-1$. Then $(fg)' = f'g + fg'$ gives

$$(x^N)' = (x^{N-1})' \cdot x + x^{N-1} \cdot x' = (N-1) x^{N-2} \cdot x + x^{N-1} = N x^{N-1}.$$

The proof is complete. □

Date: August 20, 2019.

Exercise: Why we have $(fg)' = f'g + fg'$?

Now

$$u' = \lambda u \Leftrightarrow \lambda a_n = (n+1)a_{n+1},\ n = 0, 1, \cdots .$$

Thus

$$a_{n+1} = \frac{\lambda a_n}{(n+1)} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n\cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!},$$

where we define

$$n! = 1 \cdot 2 \cdots n.$$

Then we have

$$u(x) = a_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots).$$

Put

$$E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots .$$

Since for every $C > 0$,

$$\lim_{n\to\infty} \frac{C^n}{n!} = 0,$$

we know that $E(x)$ converges for all x in $\mathbb{C}$.

Theorem 1.2. *$u(x) = E(\lambda x)$ is a unique solution of the eigenvalue equation*

$$u' = \lambda u,$$

with initail condition $u(0) = 1$.

Definition 1.3. *We shall define*

$$e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots .$$

1.2. **Definition of the exponential function.** Let us write

$$e^2 = e \cdot e,\ e^3 = e^2 \cdot e,$$

and define e^m inductively by

$$e^{n+1} = e^n \cdot e.$$

Since e is positive, we can take the $q - th$ root of e^m, we write it as $e^{\frac{m}{q}}$. Thus for every $x \in \mathbb{Q}$, e^x is well defined. The following lemma tells us that $E(x)$ is an extension of e^x from $\mathbb{Q}$ to $\mathbb{C}$.

Lemma 1.4. *For every $x \in \mathbb{Q}$, we have $e^x = E(x)$.*

Proof. Since $E(1) = e$, it suffices to prove

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \tag{2}$$

for every λ_1, λ_2 in $\mathbb{C}$. Notice that

$$(E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)' E(\lambda_2 x) + E(\lambda_2 x)' E(\lambda_1 x).$$

Put

$$G(x) = E(\lambda_1 x)E(\lambda_2 x).$$

Apply $E(\lambda x)' = \lambda E(\lambda x)$, we get

$$G' = (\lambda_1 + \lambda_2)G.$$

Notice that $G(0) = 1$. Thus Theorem 1.2 implies that

$$G(x) = E((\lambda_1 + \lambda_2)x).$$

Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. □

Exercise: Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 1.2.

Definition 1.5. *We shall use the same symbol e^x to denote $E(x)$ for all x in $\mathbb{C}$ and call e^x the **exponential function**. If $x > 0$ then we define $\ln x$ as the unique real solution of $e^{\ln x} = x$.*

By Theorem 1.2, we know that e^x is fully determined by

$$(e^x)' = e^x, \ e^0 = 1.$$

1.3. **Definition of π and trigonometric functions.** : Fix $P_0 = (1, 0)$ in the unit circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A counterclockwise rotation of P_0 gives a arc P_0P. The length, say $\theta(P)$, of the arc P_0P is a function of P. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

Definition 1.6 (Definition of π)**.** *We shall write the circumference diameter ratio as π.*

Denote by

$$F : \theta(P) \mapsto P,$$

the inverse function of $0 \leq \theta(P) \leq 2\pi$.

Definition 1.7. *We shall write $F(\theta) = (\cos\theta, \sin\theta)$.*

Notice that

$$F(0) = (1, 0) = F(2\pi), \ F(\pi) = (-1, 0), \ |F(\theta)| \equiv 1.$$

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \ \cos(0) = \cos(2\pi) = 1.$$

By definition of θ, we have

$$\int_0^{\hat{\theta}} |F'(\theta)| \, d\theta = \hat{\theta}, \ 0 \leq \hat{\theta} \leq 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$

Now $F(\theta) \cdot F(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (-\sin\theta, \cos\theta), \ \text{or } F'(\theta) = (\sin\theta, -\cos\theta).$$

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (-\sin\theta, \cos\theta),$$

which is equivalent to

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

Notice that $\cos 0 + i\sin 0 = 1$, thus Theorem 1.2 (for $\lambda = i$) gives

Theorem 1.8 (Euler's formula)**.** $e^{i\theta} = \cos\theta + i\sin\theta$.

Take $\theta = \pi$, we get the following Euler's identity

$$e^{i\pi} = -1.$$

Moreover, apply (2), we get

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)},$$

thus by Euler's formula, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2),$$

i.e.

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2, \tag{3}$$

and

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2. \tag{4}$$

Applications: Notice that n-th power of $e^{i\pi} = -1$ gives

$$e^{in\pi} = (-1)^n,$$

which implies

$$\cos n\pi = (-1)^n, \quad \sin n\pi = 0.$$

Exercise: Use $e^{i\pi/2} = i$ to compute $\cos(n\pi/2)$ and $\sin(n\pi/2)$.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: xu.wang@ntnu.no