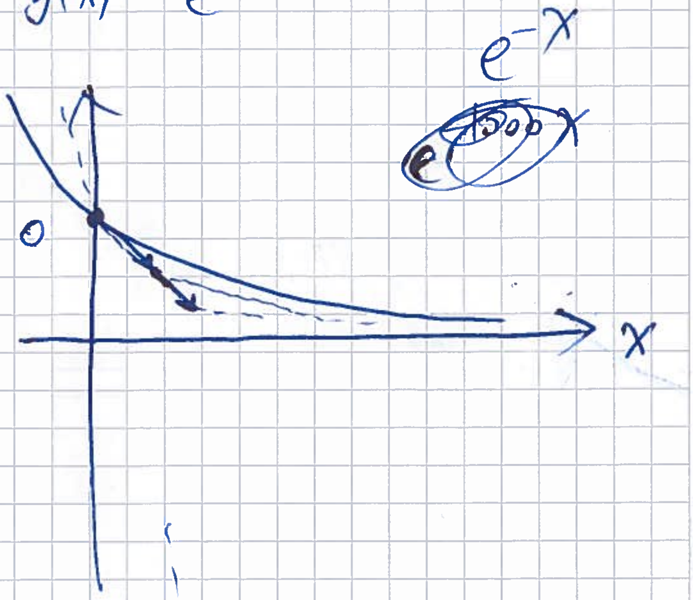
① **Example**: $\lambda = -1000$ $\Rightarrow$ $y(x) = e^{-1000x}$

$y_0 = 1$



the solution is positive

go to zero quickly.

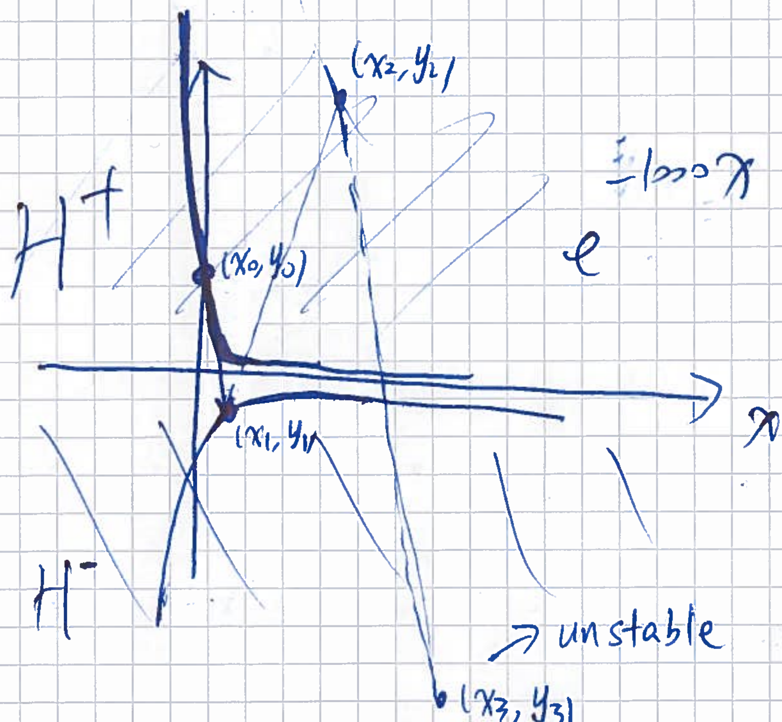
But

$$y_n = (1+\lambda h)^n y_0$$
$$= (1-1000h)^n$$

if $h = 0.01$

$$y_n = (1-10)^n = (-9)^n$$
$$\to \infty \,!$$

(But we expected $y_n \to 0$)



If we want $y_n = (1+\lambda h)^n \to 0$, need

$$-1 < 1+\lambda h < 1 \iff 0 < h < \frac{2}{-\lambda}$$

$$-1 < 1-1000h < 1$$
$$\iff \underbrace{0 < h < \frac{2}{1000}}$$

Summary: for $\begin{cases} y' = \lambda y \\ y(0) = 1 \end{cases}$ Euler method works only if the step size h satisfies $0 < h < \frac{2}{-\lambda}$

h is very small when $(-\lambda)$ is large.

__Def__. We call $y' = \lambda y$ a stiff ODE if $(-\lambda)$ is large enough!

__Example.__ Heun's method for $y' = \lambda y$

$$k_1 = f(x_n, y_n) = \lambda y_n$$

$$k_2 = f(x_n + h, y_n + hk_1) = \lambda(y_n + h\lambda y_n)$$

$$y_{n+1} = y_n + h(k_1 + k_2)/2 = y_n + \frac{h}{2}(\lambda y_n + \lambda y_n + h\lambda^2 y_n)$$

$$= y_n\left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right)$$

Put $R_H(z) = 1 + z + \frac{z^2}{2} \Rightarrow y_{n+1} = y_n \cdot R_H(\lambda h)$

__Fact:__

Apply Runge-Kutta to $y' = \lambda y$

$\Downarrow$ $\quad \exists$ polynomial $R_{RK}(z) := 1 + a_1 z + \dots + a_q z^q$.

$\underline{y_{n+1} = y_n \cdot R_{RK}(\lambda h)}$ (See Ex: 11 $\boxed{3}$).

__Def__: $S := \{ z \in \mathbb{R} : |R_{RK}(z)| \le 1 \}$

the stability interval of the corresponding Runge-Kutta method.

__Ex__. For Heun's method

$$R(z) = 1 + z + \frac{z^2}{2}$$

$$\left|1 + z + \frac{z^2}{2}\right| \le 1$$

$$\iff$$

$$-2 \le z + \frac{z^2}{2} \le 0$$

$$z^2 + 2z \le 0 \iff -2 \le z \le 0$$

$$z^2 + 2z + 4 \ge 0 \quad ✓$$

Put $z = \lambda h$

we know Heun's method works when $z \in S \iff$

$$-2 \le \lambda h \le 0$$
$$\iff \quad 0 \le h \le \frac{2}{-\lambda}$$

($-\lambda$ big : need small h)

Def: A method is $A(0)$-stable if the corresponding $R(z)$ satisfies

$$|R(z)| \le 1 \quad \text{for all } z \le 0 \,!$$

Remark 1: $A(0)$-stable $\iff S \supset [-\infty, 0]$

Remark 2: $A(0)$-stable means : if $\lambda h < 0$ then $|R(\lambda h)| \le 1$

$$\lambda h < 0 \overset{\text{for } \lambda < 0}{\iff} h > 0$$

~~thus for any big~~ ~~we~~ thus the method work for any small h !

Example: Backward Euler method. ($\to$ $A(0)$ stable!)

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$
$$= y_n + h\lambda y_{n+1}$$
$$\Downarrow$$
$$y_{n+1} = \frac{1}{1-\lambda h} y_n \Rightarrow R(z) = \frac{1}{1-z}$$

$$\left|\frac{1}{1-z}\right| \le 1 \iff |1-z| \ge 1$$
$$\iff \quad z \le 0 \text{ or } z \ge 2$$

$$\Rightarrow S = [-\infty, 0] \cup [2, \infty]$$
$$\supset [-\infty, 0]$$

thus backward Euler is $A(0)$-stable.

Stability analysis for system of ODEs

Example $\vec{y}\,' = A\vec{y}$, $a > 0$.

$$A = \begin{pmatrix} -2 & 1 \\ a-1 & -a \end{pmatrix} .$$

Apply Euler

$$\vec{y}_{n+1} = \vec{y}_n + h \cdot A \vec{y}_n$$

$y_n =$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + h \begin{pmatrix} -2 & 1 \\ a-1 & -a \end{pmatrix} \right) \cdot \vec{y}_n$$

Idea: If Eigenvalue of $\begin{pmatrix} -2 & 1 \\ a-1, & -a \end{pmatrix}$ is λ_1, λ_2

then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + h \begin{pmatrix} -2 & 0 \\ a-1, & -a \end{pmatrix}$ is $1+h\lambda_1,\ 1+h\lambda_2$

Assume that $\lambda_1 < 0$, $\lambda_2 < 0$ then "Euler" works

iff $\begin{cases} |1+h\lambda_1| \leq 1 \\ |1+h\lambda_2| \leq 1 \end{cases}$

Need to find eigenvalues of $A = \begin{pmatrix} -2 & 1 \\ a-1 & -a \end{pmatrix}$

$$\det(A - \lambda Id) = \det \begin{pmatrix} -2-\lambda & 1 \\ a-1 & -a-\lambda \end{pmatrix} = (\lambda + a)(\lambda+2) - (a-1)$$
$$= \lambda^2 + (a+2)\lambda + a + 1$$

$= (\lambda + (a+1))(\lambda + 1) \Rightarrow \lambda_1 = -1, \quad \lambda_2 = -(a+1)$

why stability is related to eigenvalues?

since $\exists\ V$ s.t.

$$V^{-1} A V = \begin{pmatrix} -1 & 0 \\ 0 & -(a+1) \end{pmatrix}$$

thus $\quad \vec{y}\,' = A \vec{y}$

$$\Updownarrow$$

$$(V^{-1}\vec{y})' = V^{-1} A V V^{-1} \vec{y}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -(a+1) \end{pmatrix} (V^{-1}\vec{y}) \quad (*)$$

Put $V^{-1}\vec{y} = \vec{z}$, then

$$\vec{z}\,' = \begin{pmatrix} -1 & 0 \\ 0 & -(a+1) \end{pmatrix} \vec{z}$$

$$\Downarrow$$

$$\begin{cases} z_1' = -z_1 \\ z_2' = -(a+1) z_2 \end{cases}$$

Euler method works for $y' = \lambda y \iff |1 + \lambda h| < 1$

$$\Downarrow$$

$$\begin{cases} |1-h| < 1 \iff 0 < h < 2 \\ |1-(a+1)h| < 1 \iff (a+1)\cdot h < 2 \end{cases}$$

Exercise. Verify that trapezoidal rule is A(?)-stable

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right).$$