Problem 1 Let u be the Heaviside function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

a) Show that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}, \quad \text{for } a \geq 0.$$

b) Solve the initial value problem

$$y''(t) + y(t) = u(t-1) \qquad y(0) = y'(0) = 0$$

and sketch its solution.

Problem 2 *This one has a weight of one task.*

a) Find the fourier series of the function

$$f(x) = \sin(3x) + \sin(x) + 1$$

on the interval $[-\pi, \pi]$.

b) ONLY 4N: Find the fourier transform of $f(x) = 6x\exp(-5x^2)$.

c) ONLY 4D: Compute the gradient of the function

$$u(x, y) = xy + y^2 + e^{2x} + \sin y.$$

Problem 3

Solve the wave equation

$$u_{tt} = u_{xx},$$

for $0 \leq x \leq \pi$ and $t \geq 0$, with boundary conditions

$$u(0, t) = u(\pi, t) = 0$$

and initial conditions

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 0.$$

Problem 4 Derive the solution formula

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) e^{\frac{-(x-v)^2}{4c^2 t}} dv, \quad t > 0,$$

for the heat equation

$$u_t = c^2 u_{xx},$$

on the whole x-axis with initial condition

$$u(x,0) = f(x).$$

Problem 5 Find the third order polynomial that interpolates $f(x) = e^x$ in $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

Problem 6 Show that

$$\frac{f(x+h) - f(x) + f(x-h)}{h^2}$$

is a second order approximation for $f''(x)$. *Hint:* Use taylor series. You may assume that f is sufficiently smooth.

Problem 7 Let $Q[f]$ be a quadrature rule that estimates the integral

$$I[f] = \int_a^b f(x)dx.$$

We know the following about this rule: there exist $s \in (a,b)$ such that

$$I[f] - Q[f] = -\frac{(b-a)^5}{6480} f^{(4)}(s).$$

Explain what a quadrature rule's degree of precision is, and find the degree of precision for this rule.

Problem 8

a) Consider the initialvalue problem

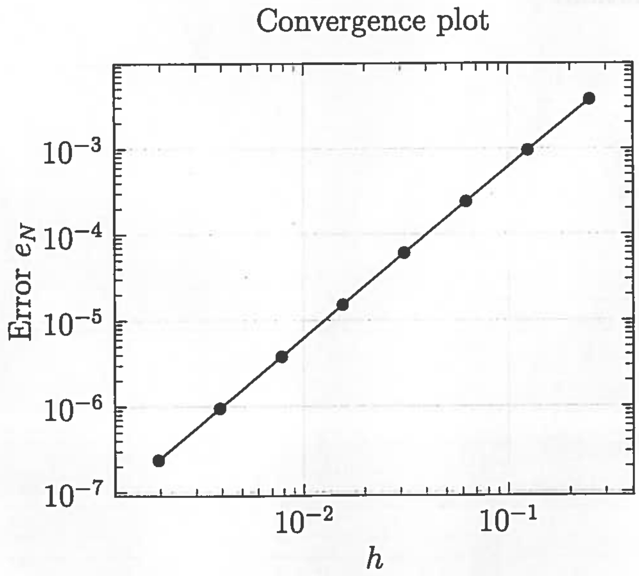
$$y' = \sqrt{y}, \quad y(0) = 1.$$

Write down a *complete* algorithm for estimating $y(2)$ by the implicit (backwards) Euler method, with step size $h = 1/N$.

Compute the first step of the algorithm with $h = 0.1$, ie. find an approximation for $y(0.1)$.

NB! The algorithm should be written in MATLAB or Python code. Include enough details that the code will run.

b) We now assume that the initial value problem is solved by an unspecified method. The error $e_N = |y(2) - y_N|$ is measured for different step sizes $h = 2/N$, and the result presented in the following convergence plot:



What is meant by the order of a method, and how may this order be read off of the plot? What is the order of this method?

Problem 9 We will solve the heat equation

$$u_t = u_{xx},$$

for $0 \leq x \leq 1$ and $t \geq 0$ with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial conditions

$$u(x,0) = x - x^2$$

Write down a complete algorithm that solves the problem with an explicit scheme for $t \in [0,1]$. Use step sizes $h = 1/M$ and $k = 1/N$ in $x-$ and $t-$directions, respectively.

Let $h = 0.2$ and $k = 0.02$ and find an approximation to $u(0.4, 0.02)$.

Assume you run your algorithm for $h = k$. What kind behaviour will you expect from the numerical solution?