Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
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Exercise set 6

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Different from the convolution in Laplace transform, the convolution in Fourier transform is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(u)g(x-u)\,du. \tag{1}$$

We have the following *Fourier convolution formula*

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f) \cdot \mathcal{F}(g), \tag{2}$$

where $\mathcal{F}(f)$ denotes the Fourier transform of f.

1 Use (2) to find the solution f for

$$\int_{-\infty}^{\infty} f(p) \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-p)^2}{2}}\,dp = e^{-\frac{x^2}{4}}.$$

2 Solve the following wave equation

$$u_{tt} = u_{xx},$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \quad \forall\, t \geq 0;$$

and initial conditions

$$u(0,x) = \sin x, \;\; u_t(0,x) = \sin 3x, \quad \forall\, 0 \leq x \leq \pi.$$

3 Solve the following heat equation

$$u_t = \frac{1}{2} u_{xx},$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \quad \forall\, t \geq 0;$$

and initial conditions

$$u(0,x) = f(x), \quad \forall\, 0 \leq x \leq \pi,$$

where

$$f(x) = x, \quad 0 \leq x \leq \frac{\pi}{2}, \quad f(x) = \pi - x, \quad \frac{\pi}{2} \leq x \leq \pi.$$

4 This exercise is on the convolution solution of the heat equation. Assume that $f \in \mathcal{S}$ (see Exercise set 5 for the definition). Let us define

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}(f \star e^{\frac{-x^2}{2t}}) = \int_{-\infty}^{\infty} f(p)\frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp.$$

a) Verify directly that u satisfies the heat equation

$$u_t = \frac{1}{2}u_{xx}.$$

b) Recall that we have (see page 17 of the notes)

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi},$$

use it to prove that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp \equiv 1.$$

c) ($**$, hard) Use a) and b) to prove that

$$\lim_{n\to\infty} u(1/n, x) = f(x).$$

Central limit theorem: The probabilistic content of the central limit theorem is as follows: Think of an infinite number of statistically independent copies $u_1, u_2, \cdots, u_n, \cdots$ of a statistical quantity u distributed according to the rule

$$P(a \le u < b) = \int_a^b f(x)\,dx,$$

in which the letter P stands for the probability of the indicated event. The adjective "independent" means that probabilities multiply:

$$P(a_j \le u_j < b_j, j = 1,2,\cdots) = \int_{a_1}^{b_1} f(x)\,dx \cdot \int_{a_2}^{b_2} f(x)\,dx \cdots$$

and you infer that the sum $s_n = u_1 + \cdot + u_n$ is a distribution according to the rule

$$P(a \le s_n < b) = \int_{a \le x_1 + \cdots + x_n < b} f(x_1)\cdots f(x_n)\,dx_1 \cdots dx_n.$$

The content of the central limit theorem is now seen to be that the scale sum $s_n/\sqrt{n}$ is nearly Gaussian distributed for large n (this is also called $\sqrt{n}$-law):

$$P(a \le \frac{s_n}{\sqrt{n}} < b) \text{ is approximately } \int_a^b (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\,dx.$$

if the density function f of the event u is normalized in the following sense

$$\int_{\mathbb{R}} x f(x)\,dx = 0, \quad \int_{\mathbb{R}} x^2 f(x)\,dx = 1.$$

The above fact goes back to Moivre and Laplace in the 18th century. The followings are the main steps of its proof.

Step 1: Show that (you may try the $n=2$ case first)

$$\int_{a\le x_1+\cdots+x_n<b} f(x_1)\cdots f(x_n)\,dx_1\cdots dx_n = \int_a^b f_n(x)\,dx,$$

where f_n is defined inductively using convolutions

$$f_1 := f, \quad f_2 := f*f, \quad f_{n+1} := f*f_n.$$

Notice that *Step 1* gives

$$P(a\le \frac{s_n}{\sqrt{n}} < b) = \sqrt{n}\int_a^b f_n(\sqrt{n}x).$$

Thus it suffices to show that $n\to\infty$ limit of $\sqrt{n}f_n(\sqrt{n}x)$ is equal to $(2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}$. This is not easy, but the Fourier transform side (*Step 2* below) is much easier!

Step 2: Show that

$$\lim_{n\to\infty}\mathcal{F}\left(\sqrt{n}f_n(\sqrt{n}x)\right) = \mathcal{F}\left((2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}\right) = (2\pi)^{-\frac{1}{2}}e^{-\frac{\omega^2}{2}}.$$

To prove it, we will use the Fourier convolution formula repeatedly, which gives (try!)

$$\mathcal{F}\left(\sqrt{n}f_n(\sqrt{n}x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}\hat{f}(\omega/\sqrt{n})\right)^n.$$

Now it suffices to show

$$\left(\sqrt{2\pi}\hat{f}(\omega/\sqrt{n})\right)^n = \left(\int_{\mathbb{R}} f(x)e^{-ix\omega/\sqrt{n}}\,dx\right)^n \to e^{-\frac{\omega^2}{2}} \tag{3}$$

as n goes to infinity. But notice that $\int f = 1$ and our "normalization assumptions" $\int xf = 0$, $\int x^2 f = 1$ imply

$$\int_{\mathbb{R}} f(x)e^{-ix\omega/\sqrt{n}}\,dx = \int_{\mathbb{R}} f(x)\left(1 - ix\omega/\sqrt{n} + \frac{(-ix\omega/\sqrt{n})^2}{2} + \cdots\right)dx = 1 - \frac{\omega^2}{2n} + \cdots.$$

Thus (3) follows from the following known identity ($\lim_{n\to\infty}(1+t/n)^n = e^t$)

$$\lim_{n\to\infty}\left(1-\frac{\omega^2}{2n}\right)^n = e^{-\frac{\omega^2}{2}}.$$

Extra exercise: not related to the exam: Try to verify

$$\lim_{n\to\infty}(1+t/n)^n = e^t$$

for every complex number t.