Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
Fall 2019

Exercise set 5

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The main theorem of Fourier transform (see page 15 of the notes) is a result on the space of smooth and rapidly decreasing functions. This space is usually called the *Schwartz space* and is denoted by $\mathcal{S}$. More precisely, we say that $f \in \mathcal{S}$ if

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \quad \text{for every } k, l \geq 0, \tag{1}$$

where $f^{(l)}$ denotes the l-th derivative of f. It is known that $x^n e^{-x^2} \in \mathcal{S}$ for every $n = 0, 1, 2, \cdots$. In the followings exercise, you might try to verify the $n = 0, 1$ cases.

1 Verify that $f_1(x) := e^{-x^2}$ and $f_2(x) := xe^{-x^2}$ satisfies the inequality (1).

2 Compute Fourier transforms of the following functions

a)

$$f(x) = \begin{cases} 1 - x^2 & -1 \leq x \leq 1 \\ 0 & |x| > 1; \end{cases}$$

b)

$$f(x) = \begin{cases} T + x & -T \leq x < 0 \\ T - x & 0 \leq x \leq T \\ 0 & |x| > T. \end{cases}$$

3 We know that the standard Gaussian function $f(x) = e^{-\frac{x^2}{2}}$ is a fixed point of the Fourier transform (see page 16 of the notes).

a) Use the above fact to compute the Fourier transform of

$$g(x) := e^{-\frac{x^2}{2}\cdot t},$$

where $t > 0$ is a constant.

b) $*$ (Difficult, not related to the exam) Prove the following *theta-identity*

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} t^{-1/2} e^{-\pi n^2/t}, \quad \forall\, t > 0.$$

c) $*$ (Difficult, not related to the exam) Try to compute the following integral

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

without using the Fourier inversion formula!

4 This exercise is on the *Heisenberg uncertainty principle.* The mathematical thrust of the principle can be formulated in terms of a relation between a function and its Fourier transform. The basic underlying law, formulated in its vaguest and most general form, states that a function and its Fourier transform cannot both be essentially localized. Somewhat more precisely, if the "preponderance" of the mass of a function is concentrated in an interval of length L, then the preponderance of the mass of its Fourier transform cannot lie in an interval of length essentially smaller than L^{-1}. The exact statement is as follows.

Heisenberg uncertainty principle: Assume that $f \in \mathcal{S}$ (see the very beginning) and $\int_{\mathbb{R}} |f(x)|^2\, dx = 1$. Then

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2\, dx \right) \cdot \left(\int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2\, d\omega \right) \geq \frac{1}{4}.$$

a) Show that if $f \in \mathcal{S}$ then integration by parts gives

$$\int_{\mathbb{R}} |f(x)|^2\, dx = -\int_{\mathbb{R}} x f'(x)\overline{f(x)}\, dx - \int_{\mathbb{R}} x f(x)\overline{f'(x)}\, dx.$$

b) Write $(f, g) := \int_{\mathbb{R}} f\bar{g}\, dx, \quad ||f||^2 := (f, f)$, show that

$$|(g, h)| \leq ||g|| \cdot ||h||.$$

Hint: use the fact that $A(t) := ||g + th||^2 \geq 0$ *for all* t *and look at the coefficient of the function* $A(t)$.

c) Use a) and b) to prove that if $f \in \mathcal{S}$ then

$$||f||^2 \leq 2 \cdot ||xf|| \cdot ||f'||.$$

d) Use the Plancherel identity to prove that

$$||f'|| = ||\omega \hat{f}||.$$

Finally, use c) to prove the Heisenberg uncertainty principle.

Recommended reading: Again recall that the standard Gaussian function $f(x) = e^{-\frac{x^2}{2}}$ is a fixed point of the Fourier transform. A natural question is

Does Fourier transform have other fixed points ?

The answer is yes! In fact it is possible (not hard acturally!) to find all the eigenvectors (the eigenvalues of the Fourier transform must be in $\{1, -1, i, -i\}$, try!) of the Fourier transform. If you are interested, I have a related note (enough to read the first two pages): *https://wiki.math.ntnu.no/_media/tma4170/2019v/fourier_2.pdf*