Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 4

Submission deadline: 1th October

1 The aim of this exercise is to show you the fundamental fact behind the *real* Fourier series: $\{1, \cos nx, \sin nx\}_{n\geq 1}$ is an orthogonal system with respect to the following inner product

$$(f, g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

a) Show that

$$(1, \sin nx) = 0, \quad (\sin nx, \cos mx) = 0, \quad \forall\ n, m = 1, 2, \cdots .$$

b) Show that

$$(\sin mx, \sin nx) = (\cos nx, \cos mx) = \begin{cases} 0 & n \neq m = 0, 1, 2, \cdots \\ \frac{1}{2} & n = m = 1, 2, \cdots . \end{cases}$$

c) Assume that

$$f = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

use a) and b) to prove

$$a_0 = (f, 1), \quad a_n = (f, 2\cos nx), \; b_n = (f, 2\sin nx)$$

and the following identity

$$||f||^2 := (f, f) = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

2 Draw the graph of the odd extension f_o and the even extension f_e of

$$f(x) = \begin{cases} x, & 0 < x \leq \frac{\pi}{2}; \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Verify the following Fourier cosine series expansion of f_e

$$f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left(-\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots\right)$$

and the Fourier sine series expansion of f_o

$$\begin{aligned} f_o(x) &= \left(\frac{2}{\pi}+1\right)\sin x + \left(0-\frac{1}{2}\right)\sin 2x + \left(\frac{-2}{3^2\pi}+\frac{1}{3}\right)\sin 3x \\ &\quad + \left(0-\frac{1}{4}\right)\sin 4x + \left(\frac{2}{5^2\pi}+\frac{1}{5}\right)\sin 5x + \cdots . \end{aligned}$$

Further notes: We know the Fourier series comes from the finite Fourier transform. Now we will show that *how to use the Fourier series to get the usual Fourier transform* (thus the usual Fourier transform also comes from the finite Fourier transform).

The *main idea* is to consider the following T-periodic function

$$f_T(x) := \sum_{k\in\mathbb{Z}} f(x+kT).$$

Similar as $T = 2\pi$ case, we can write the above T-periodic function f_T as

$$f_T(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n \frac{x}{T}}, \quad c_n := \frac{1}{T}\int_{-T/2}^{T/2} f_T(y) e^{-2\pi i n \frac{y}{T}}\, dy.$$

Notice that

$$\frac{1}{T}\int_{-T/2}^{T/2} f_T(y) e^{-2\pi i n \frac{y}{T}}\, dy = \sum_{k\in\mathbb{Z}} \frac{1}{T}\int_{-T/2}^{T/2} f(y+kT) e^{-2\pi i n \frac{y}{T}}\, dy = \frac{1}{T}\int_{\mathbb{R}} f(y) e^{-2\pi i n \frac{y}{T}}\, dy.$$

We call

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(y) e^{-2\pi i \omega y}\, dy$$

the *Fourier transform* of f (the definition here is slightly different from the notes!). Then

$$c_n = \frac{1}{T}\hat{f}(n/T),$$

thus the Fourier series of f_T gives the following *Poisson summation formula*

$$f_T(x) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}(n/T) e^{2\pi i n \frac{x}{T}}. \tag{1}$$

Assume that $f(x)$ goes to zero very fast when $|x|$ goes to infinity, then we have

$$f(x) = \lim_{T\to\infty} f_T(x) = \lim_{T\to\infty} \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}(n/T) e^{2\pi i n \frac{x}{T}} = \int_{\mathbb{R}} \hat{f}(\omega) e^{2\pi i x\omega}\, d\omega,$$

which is precisely the famous *Fourier inversion formula*! Similarly the $T \to \infty$ limit of the Parseval identity associated to (1) gives the following *Plancherel identity*

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\hat{f}(\omega)|^2\, d\omega.$$