Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
Fall 2019

Exercise set 2

Submission deadline: 17th September

1 Find the functions that have the following Laplace transforms:

a) $\frac{1}{s^2(s^2+1)}$;

b) $\frac{s}{s^2+2s+1}$;

c) $\frac{2s}{(s^2+1)^2}$;

d) $(s-3)^{-5}$.

2 Find the Laplace transforms of the following functions:

a) $f(t) = (u(t) - u(t-\pi))\cos t$;

b) $f(t) = u(t-3)t^4$.

3 Solve

$$y'' + 2y = \delta(t-1), \quad y(0) = y'(0) = 0.$$

4 By definition, a function f has period T, $T > 0$, if $f(t+T) = f(t)$. The following exercise gives you a nice Laplace transform formula for T-periodic functions.

a) Show that

$$\int_{nT}^{(n+1)T} e^{-st} f(t)\, dt = e^{-snT} \int_0^T e^{-st} f(t)\, dt$$

for T-periodic f.

b) Show that if f has period T then its Laplace transform F satisfies:

$$F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\, dt.$$

c) Show that if f has period T then its Laplace transform F satisfies:

$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0.$$

Hint: use (proofs are given below)

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad \forall\, 0 < a < 1. \tag{1}$$

Remark: The proof of (1) is based on a simple but very useful formula

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}, \quad \forall\, a \in \mathbb{C}, \quad a \neq 1. \tag{2}$$

In fact, put

$$I := 1 + a + \cdots + a^N.$$

Then of course

$$aI = a + a^2 + \cdots + a^{N+1}.$$

Thus the difference is given by

$$I - aI = 1 - a^{N+1},$$

which implies that

$$\sum_{n=0}^{N} a^n = I = \frac{1-a^{N+1}}{1-a}$$

for $a \neq 1$. Let N go to infinity, the above formula gives (1).

Application of (2) **in finite Fourier analysis**: The finite Fourier transform of a discrete signal (think of it as a vector) of length N, say

$$v = (v_0, \cdots, v_{N-1}) \in \mathbb{C}^N,$$

is given by another length N signal

$$\hat{v} = (\hat{v}_0, \cdots, \hat{v}_{N-1}),$$

where

$$\hat{v}_j := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} v_k, \quad \omega := e^{2\pi i/N}.$$

One may use (2) to prove the following

Fact: one may fully recover a signal from its Fourier transform.

More precisely, (2) implies the following *Fourier inversion formula*

$$v_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} \hat{v}_k.$$

Below is the proof: notice that

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} \hat{v}_k = \frac{1}{N} \sum_{p,k=0}^{N-1} \omega^{jk} \omega^{-kp} v_p = \frac{1}{N} \sum_{p,k=0}^{N-1} \omega^{k(j-p)} v_p.$$

It is enough to show

$$\sum_{k=0}^{N-1} \omega^{k(j-p)} = \begin{cases} N & \text{if } p = j \\ 0 & \text{if } p \neq j. \end{cases}.$$

The $p = j$ case is trivial since $\omega^0 = 1$; in case $p \neq j$, by (2) we have

$$\sum_{k=0}^{N-1} \omega^{k(j-p)} = \frac{1 - \omega^{N(j-p)}}{1 - \omega} = 0 \quad \text{since } \omega^N = 1.$$

You might also use (2) to prove the following fundamental *Parseval identity*

$$\sum_{k=0}^{N-1} |v_k|^2 = \sum_{j=0}^{N-1} |\hat{v}_j|^2.$$

Further readings (not assumed in this course): Think of a length $2N$ signal as a combination of two length N signals (i.e. its even and odd part). Then the level $2N$-Fourier transform reduces to two level N-Fourier transforms, this is the main idea of the famous *Fast Fourier Transform Algorithm (short for FFT).* A very nice description of FFT can be found in the following book:

E. Stein and R. Shakarchi, ***Fourier analysis****, Princeton lectures in analysis.*